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Spin, Mass, and Symmetry

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1. Introduction

When the strong interactions were a mystery, spin seemed to be just a complication on top of an already puzzling set of phenomena. But now that particle physicists have understood the strong, weak, and electromagnetic interactions to be gauge theories, with matter built of quarks and leptons, we recognize that the special properties of spin $\frac{1}{2}$ and spin 1 particles have taken central role in our understanding of Nature. The lectures in this summer school will be devoted to the use of spin in unravelling detailed questions about the fundamental interactions. Thus, why not begin by posing a deeper question: Why is there spin? More precisely, why do the basic pointlike constituents of Nature carry intrinsic nonzero quanta of angular momentum?

The nature and realization of spin is one of the deep questions in quantum field theory. The subject has great technical complication and is often relegated to technical treatises or highly specialized articles. Some detailed treatments of spin in quantum field theory are given in refs. 1–3. But, though the technical answers are often complex, the general ideas of the physics of spin are of genuine interest to those who would like to understand modern particle physics. In these lectures, I would like to give a broad-brush treatment of this subject, emphasizing its major ideas and challenging questions.

Why is there spin? Three different kinds of explanatory principles can be brought forth to answer this question. These might be called *permissive*, *a posteriori*, and *a priori* or *constructive* explanations. Some people are satisfied with an explanation at any of these levels; others will insist on the third, strongest type of explanation. Let us consider each level in turn.

A *permissive* explanation invokes the Totalitarian Principle of Physics: Whatever is allowed, must exist. Under this philosophy, we can explain spin by showing that it is a natural consequence of some general formal structure. I will review in the next section the idea, uncovered by Wigner, that the representations of the

Poincaré group include naturally include point particles with intrinsic spin. If such particles are possible, why can't they occur?

The idea of *a posteriori* explanation takes this argument a step further. At this level, one still will not claim to understand why particles have spin, but one argues that, without it, there would be a disaster. Such arguments use the Anthropic Principle that the world we see must be such that we can live in it. The Anthropic Principle gives a particularly strong case for the existence of spin: Without spin, particles would not obey the Pauli exclusion principle. But without the stability of Pauli exclusion, matter could avoid collapse only by finding delicate equilibrium states, such as that of the Wigner crystal, which become unstable at high density. Thus, it would be extremely difficult to build up the ordered assemblages of matter that are needed to make intelligent life.

Are these principles satisfying? Ultimately, this question goes beyond physics. It is possible that a Creator envisioned an ordered Universe and included the ingredients necessary to bring it about. Linde has argued for another point of view, that the universe contains as small domains regions in which the laws of physics are realized in all possible ways.^[4] Then we inhabit the domain in which we can live.

However, both of these explanatory principles seem to me much less compelling than a *constructive* principle which explains the ingredients of Nature as consequences of a grand pattern of symmetry. The constructive argument for the existence of particles with spin 1 is by now familiar to all particle physicists: If the equations of the universe possess a local gauge symmetry, then to each generator Q^a of the gauge group, there must correspond a vector field A_μ^a . The quantization of this field produces spin 1 particles. Unfortunately, there is no equally simple and compelling argument for spin $\frac{1}{2}$.

How close can we come to a constructive argument for spin $\frac{1}{2}$? Can we find a unified explanation for particles of spin $\frac{1}{2}$, spin 1, and perhaps higher spins? That is the question I will explore in these lectures. First, of all, I will build up the

basic formalism of spin. In Section 2, I will review the general principles which govern particles and fields with spin; then I will apply these principles successively, in Section 3, to spin $\frac{1}{2}$, in Section 4, to spin 1, and in Section 5, to spin $\frac{3}{2}$ and higher. With this foundation, I will turn in Sections 6–8 to the question of the origin of spin $\frac{1}{2}$, reviewing three proposals of increasing sophistication.

2. The Poincaré Group

Any formal discussion of spin must start from the representations of the Poincaré group, the fundamental spacetime symmetry group of translations and Lorentz transformations. Any object that lives in Minkowski space must belong to some representation of the Poincaré group. By constructing the simplest representations of the Poincaré group, we will find that intrinsic spin appears in a natural way.

One subtlety of this discussion will be that particles and fields transform in different representations of the Poincaré group. In elementary discussions of quantum field theory, one is taught that there is a direct correspondence between the particle and the field. For fields with spin, we will see that this correspondence is not so simple. In fact, the difficulty in finding the correspondence between particles and fields for fields of high spin will turn out to be an essential one which gives a crucial restriction on what fields can appear in Nature.

2.1. THE ROTATION GROUP

The generators of the Poincaré group are three sets of vectors, the generators of rotations, boosts, and translations. We will call these

$$J^i, \quad K^i, \quad P^i, \tag{2.1}$$

respectively. As a first step toward finding the representations of this group, we can start with a small, familiar piece, the group of rotations.

The generators of rotations obey the commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k. \quad (2.2)$$

The representations of these commutation relations are familiar from any book on nonrelativistic quantum mechanics: They are the multiplets of spin j , the states $|j, j^3\rangle$, with $j = 0, \frac{1}{2}, \dots$ and $j^3 = -j, \dots, j$.

The simplest nontrivial representations are those of with $j = \frac{1}{2}$ and $j = 1$. For $j = \frac{1}{2}$, we represent

$$J^i = \frac{1}{2}\sigma^i, \quad (2.3)$$

with σ^i a 2×2 Pauli sigma matrix. These generators, and the 2×2 rotation matrices built from them, act on 2-component spinors ξ_α , with $\alpha = +, -$ corresponding to $j^3 = +\frac{1}{2}, -\frac{1}{2}$.

For $j = 1$, the representation consists of 3-dimensional vectors v^i , and so the J^i must be represented by 3×3 matrices. For example

$$J^3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.4)$$

There is another way to describe this matrix action, as follows: Consider a system with two 2-component spinors. The state of this system is described by a tensor carrying two spinor indices, $T_{\alpha\beta}$. Any such tensor can be divided into its symmetric and its antisymmetric part. The most general 2×2 antisymmetric tensor is proportional to $\epsilon_{\alpha\beta}$; this object is invariant to spinor rotations. The remaining symmetric 2×2 tensor has 3 independent components and transforms, in fact, precisely as the 3-dimensional $j = 1$ representation of angular momentum. This decomposition of a 2×2 matrix of spinors into an invariant ($j = 0$) and a $j = 1$

multiplet is just the familiar angular momentum decomposition

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1 ; \quad (2.5)$$

you might recall that the $j = 0$ is the antisymmetric combination and the $j = 1$ is the symmetric combination of two spin $\frac{1}{2}$ systems.

This construction generalizes to any j . The multiplet of spin j can always be represented as a totally symmetric tensor with $2j$ two-component spinor indices:

$$\Xi_{\{\alpha\beta\cdots\delta\}}. \quad (2.6)$$

It is easy to check that this object has $2j + 1$ components, and that its highest values of j^3 , given by $\alpha = \beta = \cdots = \delta = +$, is $j^3 = j$. You can view (2.6) as what remains when the lower-spin components of a general tensor are projected out by contracting indices with the invariant $\epsilon^{\alpha\beta}$.

Now that we have a general picture of the representations of the rotation group, we can find the representations appropriate to particles and to fields. Particles are particular states of the Hilbert space with localized excitation; these can be classified by their values of j and j^3 :

$$|j, j^3\rangle . \quad (2.7)$$

Fields are operators which are functions of the space-time position x^μ . A general field can be written as a member of a multiplet of fields

$$\Phi_{\{\alpha\beta\gamma\}}(x) , \quad (2.8)$$

in which, one must remember, a rotation acts both on the spinor indices and on the spatial position. Usually, a field with explicit indices corresponding to spin j will create a particle of intrinsic spin j . However, this correspondence is not at all obvious, since the field (2.8) will create eigenstates of the Hamiltonian with all (half-integer) values of angular momentum. To understand the correspondence between particles and fields, we must probe more deeply.

2.2. THE LORENTZ GROUP

The next step in finding representations of the Poincaré group is to add the generators of boosts. This gives the commutation relations of the Lorentz group:

$$\begin{aligned} [J^i, J^j] &= i\epsilon^{ijk} J^k \\ [J^i, K^j] &= i\epsilon^{ijk} K^k \\ [K^i, K^j] &= -i\epsilon^{ijk} J^k. \end{aligned} \tag{2.9}$$

The appearance of J^k in the last line tells us that the composition of boosts produces a rotation; this effect is known as the *Wigner rotation*. Indeed, essentially all of the representation theory from here on was first formulated by Wigner.^[5] The minus sign in the last line of (2.9) tells us that the rotations generated by J^i and K^i are not four-dimensional rotations covering a compact space but rather are transformations which span noncompact spaces—the hyperboloids of Minkowski geometry.

There is a simple trick for finding the representations of the commutation relations (2.9). Let

$$J_{\pm}^j = \frac{1}{2}(J^j \pm iK^j). \tag{2.10}$$

Then the generators J_+^j and J_-^j commute with one another and obey the commutation relations

$$[J_{\pm}^i, J_{\pm}^j] = i\epsilon^{ijk} J_{\pm}^k \tag{2.11}$$

among themselves. These latter commutation relations are identical to the commutation relations of angular momentum. Thus, we can find representations of the original Lorentz group relations (2.9) by choosing a representation for J_+^j of definite angular momentum j_+ , choosing a representation for J_-^j of definite angular

momentum j_- , and then recombining these into J^j and K^j by inverting (2.10):

$$J^j = J_+^j + J_-^j , \quad K^j = -i(J_+^j + J_-^j) . \quad (2.12)$$

We denote this representation as (j_+, j_-) ; it is a representation of (2.9) of dimension $(2j_+ + 1) \times (2j_- + 1)$. We can write the object which transforms in this representation as a tensor

$$\phi_{\{\dot{\alpha}\dot{\beta}\dots\dot{\gamma}\}\{\zeta\eta\dots\theta\}} . \quad (2.13)$$

In general, I will place a dot over an index acted on by the generators J_+^j .

Notice the factor of i in the reconstruction of K^j in (2.12). This means that K^j will not be Hermitian, and so the representation we have constructed will not be unitary. This is the unfortunate but inevitable result of attempting to find a finite-dimensional unitary representation of a noncompact group action. Under Hermitian conjugation, $(J_+^j)^\dagger = J_-^j$; thus, the representation (j_+, j_-) is complex, with

$$(j_+, j_-)^* = (j_-, j_+) . \quad (2.14)$$

2.3. FIELDS UNDER THE POINCARÉ GROUP

The remaining generators of the Poincaré group, the translation generators P^i , commute with J^i and K^i and with each other, so it is easy to take them into account. We can now write general representations of the Poincaré group on multiplets of fields. To construct these, we set up field with the spinor indices corresponding to a representation (j_+, j_-) of the Lorentz group. We then make the field a function of x^μ , allowing rotations, boosts, and translations to have their standard action on this spacetime coordinate.

Here are some examples of this construction. In each case, I would like to indicate in particular the action of a boost in the $\hat{3}$ direction. To parametrize

boosts, I will use the rapidity y , defined by

$$e^y = \gamma(1 + \beta) = \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (2.15)$$

With this notation, a boost is represented in general as

$$\exp[i\vec{y} \cdot \vec{K}] \quad (2.16)$$

in particular, in successive boosts along the same axis, the rapidities add.

The simplest representations of the Lorentz group are those with one spinor index: $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. From the $(\frac{1}{2}, 0)$ representation, we can build a field $\psi_{\dot{\alpha}}(x)$. Under a rotation about the $\hat{3}$ axis, this field transforms as

$$\psi \rightarrow e^{i\theta\sigma^3/2}\psi, \quad (2.17)$$

that is, as a spin $\frac{1}{2}$ object. Under a boost, it transforms as

$$\psi \rightarrow e^{y\sigma^3/2}\psi; \quad (2.18)$$

this transformation increases the field amplitude if the spin is parallel to $\hat{3}$. A field in the $(0, \frac{1}{2})$ representation, $\psi_{\alpha}(x)$, has the same transformation under rotations, but the opposite transformation under boosts.

$$\psi \rightarrow e^{-y\sigma^3/2}\psi. \quad (2.19)$$

The next example is a field in the $(\frac{1}{2}, \frac{1}{2})$ representation, $V_{\alpha\dot{\alpha}}(x)$. This field transforms under rotations as $\frac{1}{2} \times \frac{1}{2} = \text{spin 0} + \text{spin 1}$. Under boosts in the $\hat{3}$ direction, the various components of $V_{\alpha\dot{\alpha}}$ transform as

$$(V_{+\dot{+}}, V_{-\dot{-}}, V_{-\dot{+}}, V_{+\dot{-}}) \rightarrow (V_{+\dot{+}}, V_{-\dot{-}}, e^y V_{-\dot{+}}, e^{-y} V_{+\dot{-}}). \quad (2.20)$$

All of these properties correspond to those of a field with a 4-vector index V^μ . Such a field transforms under rotations as a multiplet (V^0, \vec{V}) —spin 0 plus spin

1—and the combinations of components

$$(V^1 + iV^2, V^1 - iV^2, V^0 + V^3, V^0 - V^3) \quad (2.21)$$

transform under boosts according to (2.20).

The last simple representation we consider is that of a field belonging to the $(1, 0)$ representation: $\Phi_{\{\dot{\alpha}\dot{\beta}\}}(x)$. This field is complex, with its complex conjugate belonging to the $(0, 1)$ representation. The two pieces together give a structure with 6 real degrees of freedom. The two fields transform under rotations as spin 1. Under boosts the three components of Φ transform as

$$(\Phi_{++}, \Phi_{+\perp}, \Phi_{-\perp}) \rightarrow (e^y \Phi_{++}, \Phi_{+\perp}, e^{-y} \Phi_{-\perp}) \quad (2.22)$$

All of these properties accord with the identification of Φ as the combination of electromagnetic fields

$$\mathcal{E}^i = E^i + iB^i . \quad (2.23)$$

The field components

$$(\mathcal{E}^1 + i\mathcal{E}^2, \mathcal{E}^3, \mathcal{E}^1 - i\mathcal{E}^2) \quad (2.24)$$

indeed transform as (2.22). The conjugate combination of fields $\bar{\mathcal{E}}^i = E^i - iB^i$ belongs to the $(0, 1)$ Lorentz representation.

The last two examples presented familiar vector and tensor fields in a rather unfamiliar notation. To connect the formulae given here to more familiar ones, we should recall Dirac's famous trick for finding representations of the commutation relations of the Lorentz group. Dirac suggested that one find matrices which satisfy the simpler algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (2.25)$$

and form the combinations

$$\Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] . \quad (2.26)$$

Then the components

$$J^i = \frac{1}{2} \epsilon^{ijk} \Sigma^{jk}, \quad K^i = \Sigma^{oi} \quad (2.27)$$

satisfy (2.9). In 4-dimensional spacetime, the simplest representations of (2.25) are 4×4 matrices. Dirac's trick gives a representation of the Poincaré group as a 4-component field; this is the standard Dirac spinor Ψ .

A convenient explicit set of 4×4 matrices satisfying Dirac's relation (2.25) is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^{\mu*} & 0 \end{pmatrix}, \quad (2.28)$$

with the 2×2 components

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \quad (2.29)$$

In this basis, the combinations (2.10) are given by

$$J_+^i = \begin{pmatrix} 0 \\ \sigma^i/2 \end{pmatrix}, \quad J_-^i = \begin{pmatrix} \sigma^i/2 \\ 0 \end{pmatrix}. \quad (2.30)$$

Thus, the Dirac spinor Ψ is revealed to be a pair of 2-component fields which transform as a $(0, \frac{1}{2})$ and a $(\frac{1}{2}, 0)$ under the Lorentz group:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \psi_{\dot{\alpha}} \end{pmatrix}. \quad (2.31)$$

With this notation, the components of γ^μ carry the indices $\sigma_{\alpha\dot{\alpha}}^\mu$. It is quite appropriate to think of these constant matrices as the Clebsch-Gordon coefficients which link the $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$ or 4-vector representations. The two sets of σ^μ are not distinct; they are related by a similarity transformation:

$$\bar{\sigma}^\mu = \sigma^2 (\sigma^\mu)^T \sigma^2 \quad (2.32)$$

The factor σ^2 reflects the complex conjugation relation of the $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ representations. In order to build a field from ψ_α which transforms *exactly* like a

$(\frac{1}{2}, 0)$, one must change the basis for the conjugate of ψ_α according to

$$\tilde{\psi}_{\dot{\alpha}} = (\psi^\dagger \sigma^2)_{\dot{\alpha}} . \quad (2.33)$$

Using the invariant σ^μ , we can identify the fields $V_{\alpha\dot{\alpha}}$ and $\Phi\{\dot{\alpha}\dot{\beta}\}$ about with fields carrying more familiar combinations of indices. For the vector field

$$V_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu V_\mu . \quad (2.34)$$

The electromagnetic field strength is usually written as an antisymmetric tensor $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. Then we can write

$$\Phi_{\dot{\alpha}\dot{\beta}} = \sigma_{\alpha\dot{\alpha}}^\mu \sigma_{\beta\dot{\beta}}^\nu \epsilon^{\alpha\beta} F_{\mu\nu} . \quad (2.35)$$

Notice that the tensor Φ is indeed required to be symmetric as a consequence of the antisymmetry of the other elements in (2.35).

2.4. PARTICLES UNDER THE POINCARÉ GROUP

To describe the transformations of particles under the Poincaré group, we use a somewhat different language. While fields are operators which carry indices, particles are states in the Hilbert space of the quantum field theory. The transformation of a field need not be unitary, but transformations of states in the Hilbert space must be. It is thus useful to represent the various boosts and rotations of a given particle by the actions of these abstract unitary transformations.

If the particle has mass, it is most convenient to begin from its rest frame. In this frame, a particle of spin s forms a multiplet of $(2s + 1)$ states

$$|\vec{p} = 0; s s^3\rangle \quad (2.36)$$

which transform into one another under rotations. The boosts of these states can be defined as

$$|\vec{p}; s s^3\rangle = \Lambda(\vec{p}) |\vec{0}; s s^3\rangle , \quad (2.37)$$

where $\Lambda(\vec{p})$ is the unitary transformation which implements the boost.

Since boosts and rotations do not commute, we profit from being very careful in defining the order of the boosts and rotations that lead to a given states. For relativistic particles, it is often most convenient to quantize the spin along the direction of motion. In this system, states are labeled by their *helicity*, their spin projection along the direction of motion. If \hat{p} is a unit vector parallel to \vec{p} , the helicity is

$$\lambda = \vec{s} \cdot \hat{p}. \quad (2.38)$$

The wonderful properties of this representation are explained in a classic paper of Jacob and Wick.^[6] To write a state explicitly in the helicity representation, start from a specific spin state in the rest frame, boost from a rest parallel to the $\hat{3}$ axis, and then rotate to bring the momentum \vec{p} into its correct orientation. If the orientation of \vec{p} is given by polar and azimuthal angles θ and ϕ , the state of helicity λ is defined from the rest frame state by

$$|p, \theta, \phi; \lambda\rangle = e^{-i\phi J^3} e^{-i\theta J^2} e^{i\phi J^3} \Lambda(p\hat{3}) |\vec{0}; s s^3 = \lambda\rangle. \quad (2.39)$$

Notice that the helicity λ appears only in the rest frame state. Helicity is invariant under spatial rotations and under boosts parallel to the direction of motion.

The multiplet of states of the form (2.39) form a unitary representation of Poincaré group. This representation is infinite-dimensional. As we have noted, that is a necessary property if we insist that the group action is unitary. But this means that there is no automatic relations between the transformation properties of particles and fields.

The simplest way to make a correspondence between the particle and field transformations is to connect the field with the particle state that it creates or destroys. For low spin, this is straightforward. The free scalar field $\phi(x)$ creates and destroys scalar particles. The Fourier transform $\tilde{\phi}(p)$ precisely destroys particles with momentum p . For spin- $\frac{1}{2}$, there is a similar relation: the free Dirac field

$(\psi_\alpha(x), \psi_{\dot{\alpha}}(x))$ destroys spin- $\frac{1}{2}$ particles and creates their antiparticles according to relations

$$\langle 0 | \psi_\alpha(x) | p, \lambda \rangle = u_\alpha^\lambda(p) e^{-ip \cdot x} , \quad (2.40)$$

where the right-hand side is a solution to the free Dirac equation. Note that in this case half of the components correspond to particles destroyed, while the other half correspond to antiparticles created.

However, beginning with spin 1, problems arise in this identification. A free vector field $V^\mu(x)$ creates a particle polarized in the direction μ . This is confusing if $\mu = 0$, since a vector particle has only three polarization states, corresponding in the rest frame to the three spatial directions. If we had a fourth polarization state of a vector particle, its inner product with the other states would need to conform to the requirements of relativistic invariance, and we would find

$$\langle p, \mu | p', \nu \rangle = -g^{\mu\nu}\delta(p - p') . \quad (2.41)$$

This is a negative inner product—negative probability—for $\mu = \nu = 0$. This mismatch persists for the spin- $\frac{3}{2}$ field $\psi_{\mu\alpha}$, and gets worse for fields with multiple 4-vector indices.

2.5. MASSLESS PARTICLES UNDER THE POINCARÉ GROUP

The mismatch between particle and field degrees of freedom, which is already a problem for massive particles, becomes even worse for massless particles. To understand the new complication, we should think a bit more about the invariances of the helicity.

For a massive particle, helicity is not invariant to all operations of the Poincaré group. It is easy to see that the massive particle can be boosted to rest, and then boosted into any other direction, allowing an arbitrary change in its helicity (Fig. 1). However, this pathway is not available for a massless particle, which

can never be boosted to rest. In fact, for a massless particle, the helicity is a Poincaré invariant. This means that massless particles live in extremely small representations of the Poincaré group. A typical one is shown in Fig. 2. It consists of a particle in a state of definite helicity λ , which may be boosted to an arbitrary lightlike momentum, and its conjugate under CPT , which is an antiparticle of helicity $-\lambda$.

Massless particles are created and destroyed by fields which obey massless wave equations. Thus, one might ask, which component of the field creates the particle? To answer this question rigorously, one must perform a careful analysis of the field equation. Here, I will give a partial answer to the question using a shortcut which involves dimensional analysis. To begin, recall that the matrix element through which a field destroys a particle is dimensionless in the case of an integer-spin field and proportional to $|p|^{1/2}$ in the case of a half-integer spin field:

$$\langle 0 | \Phi(x) | p, \lambda \rangle = \left\{ \begin{array}{l} \epsilon \sim 1 \\ u(p) \sim |p|^{1/2} \end{array} \right\} e^{-ip \cdot x}. \quad (2.42)$$

Assume that p is parallel to the $\hat{3}$ axis. We now write the state $|p, \lambda\rangle$ as the boost $\Lambda = e^{iyK^3}$ of a state at lower momentum p' . Since the vacuum is boost invariant, we can rearrange the matrix element as follows:

$$\begin{aligned} \langle 0 | \Phi(x) | p, \lambda \rangle &= \langle 0 | \Phi(x)\Lambda | p', \lambda \rangle \\ &= \langle 0 | \Lambda^{-1}\Phi(x)\Lambda | p', \lambda \rangle . \end{aligned} \quad (2.43)$$

The dependence of the matrix element on p is now contained in the transformation law of the operator, and we can work this out using the formulae of Section 2.3.

Consider first a spinor field with an undotted index. From eq. (2.18), we can read the transformation law

$$\Lambda^{-1}\psi_\alpha\Lambda = (e^{-y\sigma^3/2})_{\alpha\beta}\psi_\beta. \quad (2.44)$$

This expression is proportional to $|p|^{-1/2}$ for $\alpha = +$, and to $|p|^{1/2}$ for $\alpha = -$. Only the second relation agrees with dimensional analysis. If the particle were massive,

the amplitude for the $\alpha = +$ component to destroy a fermion could consistently have the form $|m^2/p|^{1/2}$ at large p ; however, for a massless particle, this form is not available. We conclude that ψ_α destroys only *left-handed* massless spin- $\frac{1}{2}$ particles. By a similar argument, we would find that this field can also create their right-handed antiparticles. Since $\psi_{\dot{\alpha}}$ has just the opposite transformation property under boost, we would find that a fermion field with a dotted index is associated with *right-handed* massless fermions and left-handed antifermions.

A similar argument can be made for the matrix element for a vector field $A_{\alpha\dot{\alpha}}$ to destroy a vector particle. Applying (2.43) and the transformation law given in (2.20), we find

$$\langle 0 | A_{\alpha\dot{\alpha}}(x) | p, \lambda \rangle = \begin{cases} e^y \sim p & \alpha\dot{\alpha} = -\dot{+} \\ 1 & \alpha\dot{\alpha} = +\dot{+}, -\dot{-} \\ e^{-y} \sim p^{-1} & \alpha\dot{\alpha} = +\dot{-} \end{cases}. \quad (2.45)$$

Only the middle relation is consistent with dimensional analysis. Thus, $A_{\alpha\dot{\alpha}}$ destroys, and creates, states with helicity $\lambda = 1$ and -1 , but not $\lambda = 0$.

For higher-spin fields, this dimensional analysis argument allows more possibilities, and one must work out the explicit consequences of the equations of motion to exclude some of these. The general conclusion is that only the field components which create maximal helicity have one-particle matrix elements. For the spin-2 field, for example, the field components which create massless particles are

$$g_{+++}, \quad g_{--\dot{+}\dot{-}}. \quad (2.46)$$

These create and destroy particles of helicity ± 2 .

Up to this point, we have only addressed the question of which matrix elements can and cannot be zero, on general principles. It is a separate question to write a set of equations of motion which lead to the correct one-particle matrix elements of fields, and which give these fields a consistent set of interactions. To study that question we will consider a series of specific examples, beginning with spin $\frac{1}{2}$ and working upward.

3. Spin $\frac{1}{2}$

Systems with spin $\frac{1}{2}$ provide the simplest examples in which there is a nontrivial relationship between the quantum fields and the particles they create and destroy. Many of the complications we will find with spin 1 and higher are absent here, but nevertheless, the equations of motion of spin $\frac{1}{2}$ fields have many interesting features which are dictated by Lorentz invariance. In addition, the most important particles of the standard model—the quarks and leptons—have spin $\frac{1}{2}$, and many of the fundamental questions we have about these particles are posed most clearly in a language which appreciates the constraints given by space-time symmetries.

3.1. SPIN $\frac{1}{2}$ LAGRANGIANS

The easiest way to write a set of Lorentz-covariant field equations is to derive these equations from a Lorentz-invariant Lagrangian. It is easy to construct such Lagrangians: If we begin with fields which carry dotted and undotted spinor indices, Lorentz-invariance is guaranteed if we contract all indices of each type.

As an example, we can construct the Lagrangian for a spin $\frac{1}{2}$ field ψ_α . This Lagrangian should involve the field ψ_α , its Hermitian conjugate $\psi^\dagger_{\dot{\alpha}}$, and at least one spatial derivative ∂_μ . By using the invariant $\bar{\sigma}^{\mu\dot{\alpha}\alpha}$ to convert the vector index to spinors, we can contract all the indices by writing

$$\mathcal{L} = \psi^\dagger_{\dot{\alpha}} i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha . \quad (3.1)$$

This is the simplest possible spin $\frac{1}{2}$ Lagrangian, involving a 2-component, not a 4-component, field. In a moment, I will show how to reconstruct the familiar Dirac Lagrangian from this starting point.

The field equation following from the Lagrangian (3.1) is

$$i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha = 0 . \quad (3.2)$$

This is the *Weyl equation*. Multiplying on the left by $i\sigma_{\alpha\dot{\alpha}}^\nu \partial_\nu$ and using $\sigma^\nu \bar{\sigma}^\mu = g^{\mu\nu}$,

this equation becomes

$$\partial^2 \psi_\alpha = 0 . \quad (3.3)$$

Thus, the Weyl equation is an equation for massless particles. However, not every massless wave function satisfies (3.2). If we look for solutions to (3.2) of the form of a plane wave,

$$\psi = u(p) e^{-ip \cdot x} , \quad (3.4)$$

where $u(p)$ is a 2-component constant vector, this vector satisfies

$$\bar{\sigma} \cdot p u(p) = (p^0 + \vec{\sigma} \cdot \vec{p}) u(p) = 0 . \quad (3.5)$$

This equation implies that $u(p)$ is proportional to a 2-component spinor which is left-handed with respect to the direction of motion. If ξ_α is a spinor normalized to $\|\xi\| = 1$, then

$$\psi_\alpha = \sqrt{2E} \xi_\alpha e^{-ip \cdot x} . \quad (3.6)$$

Along with the Lagrangian (3.1), there is another equally simple Lagrangian involving the spin $\frac{1}{2}$ field with a dotted index, $\tilde{\psi}_{\dot{\alpha}}$:

$$\mathcal{L} = \tilde{\psi}_\alpha^\dagger i \sigma^{\mu \alpha \dot{\alpha}} \partial_\mu \tilde{\psi}_{\dot{\alpha}} . \quad (3.7)$$

This Langrangian implies the field equations equation similar to (3.2) with $\bar{\sigma}^\mu$ replaced by σ^μ , leading to solutions which are right-handed with respect to the direction of motion. However, this Lagrangian is not an alternative to (3.1); instead, it is identical. We may replace $\tilde{\psi}$ with ψ^\dagger according to (2.33):

$$\tilde{\psi} = (\psi^\dagger \sigma^2)^T ; \quad \tilde{\psi}^\dagger = (\sigma^2 \psi)^T . \quad (3.8)$$

Integrate by parts, and cancel the minus sign from this manipulation against the one obtained by interchanging the order of the fermion fields. Finally, use the identity (2.32). We find that (3.7) is transformed into precisely into (3.1).

Both of these forms of the Weyl Lagrangian may be compared with the standard Dirac Lagrangian which describes electrons in quantum electrodynamics:

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu (\partial_\mu + ieA_\mu) \Psi - m\bar{\Psi}\Psi . \quad (3.9)$$

Replace the four-component Dirac spinor by two two-component spinors with undotted indices:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \psi_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_{1\alpha} \\ (\psi_2^\dagger \sigma^2)_{\dot{\alpha}} \end{pmatrix} . \quad (3.10)$$

The Dirac Lagrangian is rewritten as follows:

$$\begin{aligned} \mathcal{L} = & \psi_{1\dot{\alpha}}^\dagger i\bar{\sigma}^{\mu\dot{\alpha}\alpha} (\partial_\mu + ieA_\mu) \psi_{1\alpha} + \psi_{2\dot{\alpha}}^\dagger i\bar{\sigma}^{\mu\dot{\alpha}\alpha} (\partial_\mu - ieA_\mu) \psi_{2\alpha} \\ & - im\epsilon^{\alpha\beta} \psi_{1\alpha} \psi_{2\beta} + im\epsilon^{\dot{\alpha}\dot{\beta}} \psi_{1\alpha}^\dagger \psi_{2\beta}^\dagger . \end{aligned} \quad (3.11)$$

If we ignore the terms proportional to the electron mass, the Lagrangian splits into two pieces, one for the left-handed electron and its right-handed antiparticle, and one for the left-handed positron and its antiparticle, the right-handed electron. Notice that the sign of the charge has changed in the second term precisely in accord with this interpretation. The mass term is revealed in the second line of (3.11) to be a Lorentz-invariant mixing of the left-handed and right-handed components.

The structure of eq. (3.11) is very simple; thus, it is straightforward to generalize it. In fact, we can immediately write down the most general Lagrangian for fermions interacting with vector bosons. For reasons I will discuss in the next section, vector bosons are necessarily gauge bosons and are associated with the generators of a symmetry group. If we accept this for the moment, it makes sense to represent the most general collection of fermions as a collection of two-component fields $\psi_{a\alpha}$ on which the gauge symmetries act. Write the infinitesimal form of this transformation abstractly as:

$$\psi_{a\alpha} \rightarrow (1 + i\theta^A T^A)_{ab} \psi_{b\alpha} . \quad (3.12)$$

The gauge symmetry implies that gauge fields couple to fermions through the

covariant derivative

$$D_\mu \psi_{a\alpha} = (\partial_\mu \delta_{ab} - ig_A A_\mu^A T_{ab}^A) \psi_{b\alpha} . \quad (3.13)$$

Then the most general Lagrangian for massless fermions has the form

$$\mathcal{L} = \psi_a^\dagger i\sigma \cdot D \psi_a . \quad (3.14)$$

A mass term for these fermions has the general form

$$\Delta \mathcal{L} = -i M^{ab} \epsilon^{\alpha\beta} \psi_{a\alpha} \psi_{b\beta} + h.c. \quad (3.15)$$

Notice that the product of two fermion fields is doubly antisymmetric, since it pick up a minus sign from interchanging the fermion operators and another from interchanging the indices α and β . Thus, the mass matrix M^{ab} is symmetric.

The presentation (3.14), (3.15) of the fermion Lagrangian brings us immediately to the most fundamental questions about elementary fermions. To write the kinetic energy term (3.14), we need only the most basic information about these fermions: how many are there, and how are they organized into representations of the gauge symmetry group? To write (3.15), we need to know how these fermions link up to acquire mass. Note that these linking terms often imply breaking of the underlying gauge invariance. For example, in the electron mass term in (3.11), the left-handed electron field ψ_1 is a member of weak isospin doublet, while the left-handed positron field ψ_2 is an isospin singlet. This brings us directly to the mystery of what agent breaks this symmetry in order to allow the mixing of these components.

3.2. SPIN DECOUPLING AT LOW AND HIGH ENERGY

The Weyl or Dirac Lagrangian dictates a certain relation between the spin of the fermion and its orbital motion. To understand this relation, it is useful to work through some examples of fermion motion and its influence on the fermion spin. The limit of high energy is especially simple. In this limit, the mass terms in the Lagrangian become irrelevant, and the Lagrangian decouples into terms involving fermion components of definite helicity. Notice that the coupling to vector fields separates in exactly the same way and also conserves helicity in this limit.

Another especially simple limit is that of low energy, in which the fermion's momentum is small compared to its mass. In this limit, the effects of relativity become unimportant and a fermion looks to a good approximation like a scalar particle. The spin decouples up to effects of order $1/m$. To see this explicitly, we manipulate the Dirac equation as follows: Begin from the equation of motion of the Dirac Lagrangian (3.9), in the form

$$(i\gamma \cdot D - m)\Psi = 0 , \quad \text{where} \quad D_\mu = \partial_\mu - igA_\mu . \quad (3.16)$$

Multiply on the left by $(-i\gamma \cdot D - m)$; this gives

$$\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}D_\mu D_\nu + \frac{1}{2}[\gamma^\mu, \gamma^\nu]D_\mu D_\nu + m^2 \right) \Psi = 0 . \quad (3.17)$$

Now apply (2.25) and (2.26), and simplify the second term using the antisymmetric relation

$$[D_\mu, D_\nu] = -igF_{\mu\nu} . \quad (3.18)$$

This converts (3.17) into

$$\left(D^2 - g\Sigma^{\mu\nu}F_{\mu\nu} + m^2 \right) \Psi = 0 . \quad (3.19)$$

This last equation is similar to the Klein-Gordon equation, and it is easy to infer

from it the Schrödinger equation which gives its nonrelativistic limit. The nonrelativistic Hamiltonian is

$$H = m - \frac{1}{2m}(\vec{D})^2 + gA^0 - \frac{g}{m}\vec{\sigma} \cdot \vec{B}. \quad (3.20)$$

The first term which involves the spin is also suppressed by an explicit factor of $1/m$.

To describe how the Dirac equation interpolates between these limits, we will consider a specific practical example, the quantum electrodynamics cross section for the reaction $e^+e^- \rightarrow \mu^+\mu^-$. The kinematics of the process are shown in Fig. 3. To be specific, we will consider the annihilation of a left-handed electron with a right-handed positron, assigning the collision axis in the direction of the electron motion to be the \hat{z} axis. The electron and positron produce a virtual photon with spin 1 and $J^3 = -1$ which eventually reforms into a muon pair. The differential cross section for this process is easily worked out from Feynman diagrams. I will write the result of this calculation in a suggestive notation.

In the low energy limit, the muons are produced in an S -wave. Thus, their momenta are distributed isotropically. The angular momentum of the virtual photon must be carried by the muon spins, and these are approximately decoupled from the orbital motion. Then the final muons both have spin $S^3 = -\frac{1}{2}$. In the basis of s^3 , we can write the scattering amplitude as

$$\mathcal{M}(e_L^- e_R^+ \rightarrow \mu^- \mu^+) = -2e^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.21)$$

where the rows of the matrix denote the spin components $S^3 = +\frac{1}{2}$ and $S^3 = -\frac{1}{2}$ for the μ^- and the columns denote the spin components of the μ^+ .

To discuss the transition to high energy, it is convenient to rewrite this scattering amplitude in a basis of helicity states. These are related to the states of

definite S^3 by a rotation:

$$\begin{aligned} \left| \mu^-, S^3 = -\frac{1}{2} \right\rangle &= \cos \frac{\theta}{2} \left| \mu^-, \lambda = -\frac{1}{2} \right\rangle - \sin \frac{\theta}{2} \left| \mu^-, \lambda = +\frac{1}{2} \right\rangle \\ \left| \mu^+, S^3 = -\frac{1}{2} \right\rangle &= -\cos \frac{\theta}{2} \left| \mu^+, \lambda = +\frac{1}{2} \right\rangle - \sin \frac{\theta}{2} \left| \mu^-, \lambda = -\frac{1}{2} \right\rangle . \end{aligned} \quad (3.22)$$

In the basis of helicity states, the matrix (3.21) becomes

$$\mathcal{M}(e_L^- e_R^+ \rightarrow \mu^- \mu^+) = e^2 \begin{pmatrix} -\sin \theta & (1 - \cos \theta) \\ (1 + \cos \theta) & \sin \theta \end{pmatrix} . \quad (3.23)$$

The matrix elements are proportional to the elements of the spin-1 rotation matrices $d_{\lambda\lambda'}^1(\theta)$, as required by the general results of Jacob and Wick.^[6]

The expression (3.23) can be directly compared to the high energy limit of the scattering amplitude for $e^+ e^- \rightarrow \mu^+ \mu^-$. In that limit, we find

$$\mathcal{M}(e_L^- e_R^+ \rightarrow \mu^- \mu^+) = e^2 \begin{pmatrix} \mathcal{O}(m_\mu/E) & (1 - \cos \theta) \\ (1 + \cos \theta) & \mathcal{O}(m_\mu/E) \end{pmatrix} . \quad (3.24)$$

The elements which conserve helicity have the same form as in (3.23), while the elements which violate helicity conservation go to zero as the muons become relativistic. This matrix element leads to the unpolarized cross section

$$\frac{d\sigma}{d \cos \theta} = \frac{\pi \alpha}{2s} [(1 + \cos \theta)^2 + (1 - \cos \theta)^2] \quad (3.25)$$

which is familiar from the phenomenology of $e^+ e^-$ annihilation at energies well below the Z^0 . More generally, the appearance of $(1 + \cos \theta)^2$ and $(1 - \cos \theta)^2$ angular distributions in $e^+ e^-$ annihilation display the constraint of helicity conservation at high energy and correlate angular distributions to the couplings of the various helicity states.

4. Spin 1

After this taste of the dynamics of spin $\frac{1}{2}$ fields, we move on to a discussion of spin 1. Spin 1 is the first case in which the mismatch between field components and physical particles becomes a serious problem. In this section, I would like to explain how this problem is resolved for massless and for massive fields. The explanation has a surprising number of subtleties, but it also reveals some interesting physical consequences.

4.1. QUANTUM ELECTRODYNAMICS

The most familiar spin 1 particle is the photon. Since the photon is massless, one might think that it would be especially difficult to treat in quantum field theory. And yet, all of the problems of principle of building a quantum theory of photons are automatically answered in quantum electrodynamics. Let us review how this happens.

The Lagrangian of free photons is given by the expression

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 = +\frac{1}{2}A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu)A_\nu , \quad (4.1)$$

which leads directly to Maxwell's equations. It is a standard result of undergraduate physics that the propagating solutions of Maxwell's equations satisfy

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0 . \quad (4.2)$$

A typical solution of Maxwell's equations, propagating in the $\hat{3}$ direction, is given by taking the real and imaginary parts of the relation

$$\vec{E} + i\vec{B} = (\hat{1} \pm i\hat{2})e^{-ip \cdot x} \quad (4.3)$$

with $\vec{p} \parallel \hat{3}$, $p^0 = |\vec{p}|$. These are plane waves, propagating at the speed of light, with helicity $\lambda = \pm 1$. In agreement with the dimensional analysis argument in (2.45), there is no propagating plane wave solution to Maxwell's equation with helicity $\lambda = 0$.

In quantum electrodynamics, we represent the photon field by a propagator

$$\langle A^\mu(k)A^\nu(-k)\rangle = \frac{-ig^{\mu\nu}}{k^2} \quad (4.4)$$

which apparently contains all field components. This looks paradoxical, for two reasons. First, as we have just discussed, the helicity zero components of the photon are not associated with propagating waves. Second, the expectation value in (4.4) seems to indicate an incorrect quantum mechanics. From (4.4), one can straightforwardly derive the identity

$$\sum_{\epsilon} \langle 0 | A^\mu(x) | k, \epsilon \rangle \langle k, \epsilon | A^\nu | 0 \rangle = -g^{\mu\nu} \quad (4.5)$$

for the matrix element of the vector field between one-particle states and the vacuum. If the norms of states in Hilbert space are positive, this quantity should be positive, but the $\mu = \nu = 0$ element of (4.5) is negative. We encountered this pathology earlier, in eq. (2.41), and avoided it there only by forbidding bosons with timelike polarization. However, when we work with (4.4), we must necessarily include both bosons with helicity zero and those with timelike polarization in our formalism.

Fortunately, quantum electrodynamics magically resolves both of these problems. The crucial element required is the fact that the photon field couples to a conserved current, the current j^μ of electric charge. It is important to note that Maxwell's equations would be inconsistent if the charge current were not conserved: In relativistic form, Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = ej^\nu . \quad (4.6)$$

Thus, simply by applying ∂_ν to this equation and using the fact that $F^{\mu\nu}$ is anti-symmetric, we find

$$\partial_\nu j^\nu = 0 . \quad (4.7)$$

Alternatively, one can argue that a local gauge symmetry can only be built in a theory with a perfect global symmetry.

The conservation of the current j^μ constrains the states that can be produced in quantum electrodynamics processes. To see this, consider the matrix element for single photon emission, shown in Fig. 4, and analyze this matrix element for a photon emitted parallel to the \hat{z} axis. If we pull the photon out of the vertex function, as shown in the figure, we see that it couples to the current j^μ :

$$i\mathcal{M} = i\mathcal{M}^\mu(q)\epsilon_\mu^*(q) = -ie \langle j^\mu(q) \rangle \epsilon_\mu^*(q) . \quad (4.8)$$

Current conservation imposes the condition

$$q_\mu \langle j^\mu(q) \rangle = 0 . \quad (4.9)$$

In this situation, $q^\mu = (q, 0, 0, q)$, so this relation implies $\mathcal{M}^0 = \mathcal{M}^3$. If we take account of the negative norm (4.5) of time-like polarized photon states, we find a probability of photon emission proportional to the Lorentz-invariant combination

$$|\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 . \quad (4.10)$$

Only the first two terms of this expression correspond to physical propagating photons. But we now see that the other two terms of this expression are irrelevant, since they cancel precisely due to the constraint of gauge invariance.

It is amazing that the various unphysical aspects of the formalism work together with one another to make this cancellation occur. The organizing principle is gauge invariance. In this discussion, I have made a particular choice of gauge, the Feynman gauge. With other choices, for example, the Coulomb gauge,^[7] one can work directly with a Hilbert space which contains only the physical photon degrees of freedom, at the cost of manifest Lorentz invariance.

This same cancellation mechanism also holds in non-Abelian gauge theories with massless gauge bosons. Again, it is organized by the requirement that the current associated with the gauge symmetry be (covariantly) conserved. In the

non-Abelian case, there are additional pairs of unphysical positive and negative norm states—the Fad’deev-Popov ghosts—which enter the cancellation. The detailed proof of this cancellation is rather technical; it has been set out clearly (for theorists) by Taylor^[8] and by Kugo and Ogima.^[9]

4.2. MASSIVE SPIN 1 BOSONS

The questions that we discussed in the previous section become more intricate when we consider massive spin 1 bosons. For massive particles, all (spacelike) helicity states should be physical and correspond to propagating modes. However, in a covariant formalism, the timelike component of the vector field A^μ still must create states of negative norm. How can these conflicting demands be satisfied? For simplicity, I will consider only the case of a single massive vector boson, without the complications of non-Abelian couplings.

For this case of a single massive spin 1 boson, there are two solutions known in the literature. The first is given by adding a mass term to the Lagrangian of quantum electrodynamics, to produce the Stückelberg Lagrangian,

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}m^2 A^\mu A_\mu . \quad (4.11)$$

This Lagrangian has a very simple classical theory. The field equation is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 . \quad (4.12)$$

Applying ∂_ν to this equation, we find

$$m^2 \partial_\nu A^\nu = 0 ; \quad (4.13)$$

then the timelike component of A^μ vanishes. The remaining components of A^μ

satisfy the massive field equation

$$(\partial^2 + m^2)A^\nu = 0 . \quad (4.14)$$

The propagating solutions to this equation have the form

$$A^\mu = \epsilon^\mu(p)e^{-ip\cdot x} , \quad (4.15)$$

with $p^2 = m^2$. Eq. (4.13) imposes the constraint $p \cdot \epsilon(p) = 0$. The solutions satisfying this constraint correspond to three spacelike polarizations. In the quantum theory, by an argument similar to that of (4.8), the timelike polarization is not produced from a conserved current. Unfortunately, the most familiar massive spin 1 bosons in Nature, the W and Z bosons, couple to currents such as the weak isospin current which correspond to broken symmetries. In this case, the Stückelberg strategy breaks down.

The alternative to this strategy is to construct massive spin 1 bosons from massless gauge bosons by spontaneously breaking the gauge symmetry. This strategy, which was discovered by Higgs, Kibble, Guralnik, Hagen, Brout, and Englert, is now generally known as the *Higgs mechanism*. In its simplest formulation, one would add to the Lagrangian of electrodynamics an electrically charged scalar field φ .

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + D_\mu\varphi^\dagger D_\mu\varphi - V(|\varphi|^2) , \quad (4.16)$$

where the covariant derivative D_μ is given by $D_\mu = (\partial_\mu - igA_\mu)$, as in eq. (3.13). The function V is a potential energy for the field φ . If it becomes energetically favorable for φ to obtain a vacuum expectation value

$$\langle\varphi\rangle = \frac{1}{\sqrt{2}}v , \quad (4.17)$$

then the second term in (4.16) leads to

$$D_\mu\varphi^\dagger D_\mu\varphi \rightarrow \frac{1}{2}g^2v^2A^\mu A_\mu , \quad (4.18)$$

which is a mass term with $m = gv$. In this way, we recover the Stückelberg mass

term, but in a theory with an underlying symmetry structure.

This structure will become crucial when we try to answer more detailed questions about the nature of this massive spin 1 field. Here are two: Set up the kinematics of boson emission as in the discussion of eq. (4.8), with the boson moving parallel to the $\hat{3}$ axis. In this massive case, A^3 creates physical states, so it is no longer obvious that the negative metric states created by A^0 will be exactly cancelled. How is this guaranteed? In addition, the new physical states created by A^3 have their own difficulties. If the massive boson is emitted at rest, the new states have polarization vector $\epsilon^\mu(q) = (0, 0, 0, 1)$. The boost of this vector to momentum q is

$$\epsilon_{\text{long}}^\mu(q) = \left(\frac{q}{m}, 0, 0, \frac{E}{m} \right), \quad (4.19)$$

where $E^2 = q^2 + m^2$. In the limit of high energy, the individual components of this vector become extremely large, sufficiently so, as we will see below, to cause scattering amplitudes to violate unitarity. What controls the growth of these new amplitudes?

I will now argue that underlying local gauge invariance which is present in the Higgs mechanism supplies the answers to both of these questions. To make the connection, we need one further ingredient, which is, however, a consequence of local gauge invariance. In any local field theory in which a continuous symmetry is spontaneously broken, the theory must contain a massless particle, called a *Goldstone boson*. As an example of this general principle, we might consider the scalar field in (4.16). In the above discussion, we assumed that it is energetically favorable for φ to acquire a vacuum expectation value (4.17). Since the theory is symmetric under rotation of the phase of φ , this expectation value could equally well be generated with any phase. But now consider a field configuration such that

$$\langle \varphi(x) \rangle = e^{i\alpha(x)} \frac{1}{\sqrt{2}} v. \quad (4.20)$$

The phase variation shown here could at worst cost an energy proportional to $|\vec{\nabla}\alpha|$ which vanishes in the long wavelength limit. Thus, this phase variation

corresponds to a massless field, or, after quantization, a massless particle. In the following discussion, I will denote this particle by

$$\pi = \sqrt{2} \operatorname{Im}\varphi . \quad (4.21)$$

Notice that in a gauge theory, (4.20) is a local gauge transformation, and thus the field π can be transformed away. Nevertheless, we must retain it in our covariant-gauge formalism.

An important property of a Goldstone boson is that it is created and destroyed singly by the symmetry current. In the example of φ , the electromagnetic current is

$$j^\mu = -i(\varphi^\dagger \partial^\mu \varphi - \partial^\mu \varphi^\dagger \varphi) . \quad (4.22)$$

Inserting (4.17) and (4.21) into (4.22) to determine the piece depending on one quantum field, we find

$$j^\mu = v \partial^\mu \pi + \dots \quad (4.23)$$

Then the current can create and destroy single quanta of π . The standard form for this relation is

$$\langle 0 | j^\mu | \pi(p) \rangle = -i F p^\mu ; \quad (4.24)$$

using (4.23), we can identify $F = v$ in this example. Though the symmetry associated with j^μ is spontaneously broken, the current should still satisfy the equation of motion $\partial_\mu j^\mu = 0$. Applied to (4.24), this equation implies $p^2 = 0$, which confirms that the Goldstone boson should be massless.

The presence in the theory of a Goldstone boson allows us to understand how the spin 1 particle can acquire mass compatible with current conservation. The structure of the vector boson self-energy in the theory (4.16) is shown in Fig. 5. This amplitude is actually an expectation value of two currents; thus, it should

satisfy

$$q_\mu \langle j^\mu(q)j^\nu(-q) \rangle = 0 . \quad (4.25)$$

The mass term in the Lagrangian, eq. (4.18), contributes the term

$$-ig^{\mu\nu}m^2 , \quad (4.26)$$

with $m = gv$, which does not by itself satisfy (4.25). However, because the current j^μ can create a single Goldstone boson, there is another contribution of the same order, as shown in the figure. This new contribution uses the matrix element (4.24) and contains a Goldstone boson propagator i/q^2 . The sum of these contributions is

$$\begin{aligned} & -ig^{\mu\nu}m^2 + (-igFq^\mu)\frac{i}{q^2}(igFq^\nu) \\ &= -im^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) , \end{aligned} \quad (4.27)$$

where $m = gv = gF$. This full expression satisfies (4.25). One may, in fact, turn this argument around to show that the relation

$$m = gF \quad (4.28)$$

follows from the formula (4.24), independently of the underlying Lagrangian.

Now we have all of the ingredients we need to analyze vector boson emission in a theory with the Higgs mechanism. To begin, we should write the analogue of eq. (4.10) for the theory with massive spin 1 bosons. Let ϵ_{Ti}^μ be the polarization vectors corresponding to transverse polarizations, let $\epsilon_{\text{long}}^\mu$ be the polarization vector (4.19) corresponding to longitudinal polarization, and let ϵ_t^μ be a vector equal to $(1, 0, 0, 0)$ in the rest frame which corresponds to time-like polarization. Then the probability of emitting a spin 1 boson is proportional to

$$|\epsilon_{T1} \cdot \mathcal{M}|^2 + |\epsilon_{T2} \cdot \mathcal{M}|^2 + |\epsilon_{\text{long}} \cdot \mathcal{M}|^2 - |\epsilon_t \cdot \mathcal{M}|^2 + |\mathcal{M}_\pi| . \quad (4.29)$$

The last term in the sum involves \mathcal{M}_π , the matrix element for producing a Goldstone boson. Among these five states, the first three are expected to be physical

particles. The timelike vector boson is a state of negative norm and must therefore be unphysical. The Goldstone boson is also expected to be unphysical, as explained below eq. (4.20).

A relation between the latter two production amplitudes is given by the equation of current conservation. As in Fig. 4, we can analyze a vector boson emission amplitude by pulling on the vector boson line and revealing the current to which the boson attaches. In the case of a massive boson, the result of that manipulation is shown in Fig. 6. The current can either couple directly into the emission process, or it can couple to a single Goldstone boson which in turn joins onto the emission diagram. Thus, we find

$$\langle j^\mu(q) \rangle = \mathcal{M}^\mu - igFq^\mu \frac{i}{q^2} \mathcal{M}_\pi . \quad (4.30)$$

If the current must be conserved, we must find zero when we contract q^μ with this expression. This gives the relation

$$q_\mu \mathcal{M}^\mu + gF\mathcal{M}_\pi = 0 . \quad (4.31)$$

Since $\epsilon_t^\mu = q^\mu/m$, we find

$$|\epsilon_t \cdot \mathcal{M}|^2 = |\mathcal{M}_\pi|^2 . \quad (4.32)$$

Thus, also in the case of a massive vector boson, the underlying principles of gauge symmetry and current conservation guide the cancellation of unphysical positive and negative norm states.

This argument can be pushed a bit farther to develop an additional piece of insight. Notice that the longitudinal polarization vector (4.19) satisfies

$$\epsilon_{\text{long}}^\mu = \frac{q^\mu}{m} + \mathcal{O}\left(\frac{m}{q}\right) . \quad (4.33)$$

I have already remarked that the individual components of $\epsilon_{\text{long}}^\mu$ can be extremely

large. However, we now see that

$$\epsilon_{\text{long}} \cdot \mathcal{M} \simeq \epsilon_t \cdot \mathcal{M}, \quad (4.34)$$

which is in turn related by (4.32) to the amplitude for emission of a Goldstone boson. Thus, we find the relation shown in Fig. 7, known as the *Goldstone Boson Equivalence Theorem*.^[10,11] This formula has the following physical interpretation: Through the Higgs mechanism, the vector field becomes massive by eating the the Goldstone boson. At high energy, the spontaneous symmetry breaking becomes irrelevant, and the emission amplitudes for massive bosons show their origin as a combination of transverse spin 1 and Goldstone boson emission amplitudes.

The argument for the Goldstone Boson Equivalence Theorem is given here only at the simplest level. A more careful argument is needed when several vector bosons are emitted and when loop corrections to the boson propagators are included. However, the theorem remains true in these situations. Some recent analyses which take account of these subtleties are given in refs. 12–14.

4.3. EXAMPLES FROM THE STANDARD MODEL

If the discussion of the previous section was a bit abstract, the moral of this discussion has direct application to high energy processes in the standard electroweak gauge theory. In this section, I will present two important examples.

The first of these is the theory of the top quark width. We now know that the top quark is sufficiently heavy to decay to an on-shell W boson and a bottom quark. For this two-body decay, one might roughly estimate the width as $\Gamma_t \sim (\alpha_w/4\pi)m_t$, where $\alpha_w = g^2/4\pi = \alpha/\sin^2\theta_w$. The width of the top quark eventually controls the qualitative features of top decays, so it is important to understand its magnitude. Surprisingly, this rough estimate turns out to be very naive; the true result for Γ_t grows as m_t^3 . The explanation for this change comes from the Goldstone Boson Equivalence Theorem.

Before I explain the behavior of the top quark width, let us obtain the correct result by a straightforward Feynman diagram calculation. In the standard model, the top quark width is given to leading order by the diagram of Fig. 8(a). The decay matrix element is

$$i\mathcal{M} = \frac{ig}{\sqrt{2}} \bar{u}(p_b) \gamma^\mu \frac{(1 - \gamma^5)}{2} u(p_t) \epsilon_\mu^*(q) , \quad (4.35)$$

where q is the W momentum. From here on, ignore the b quark mass. Then the square of the matrix element is

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^2}{2} [p_b^\mu p_t^\nu + p_b^\nu p_t^\mu - g^{\mu\nu} p_b \cdot p_t] \epsilon_\mu^*(q) \epsilon_\nu(q) . \quad (4.36)$$

Now sum over the three physical W polarizations, excluding the timelike polarization:

$$\sum_{\text{pol}} \epsilon_\mu^*(q) \epsilon_\nu(q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right) . \quad (4.37)$$

This gives

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^2}{2} [p_b \cdot p_t + 2 \frac{q \cdot p_b q \cdot p_t}{m_W^2}] . \quad (4.38)$$

To simplify this expression, use the kinematic relations

$$2p_b \cdot p_t = 2p_b \cdot q = m_t^2 - m_W^2 , \quad 2p_t \cdot q = m_t^2 + m_W^2 . \quad (4.39)$$

Notice that the second term in (4.38) is of order (m_t^4/m_W^2) . Add phase space factors to obtain the final result

$$\Gamma_t = \frac{g^2}{64\pi} \frac{m_t^3}{m_W^2} \left(1 - \frac{m_W^2}{m_t^2}\right)^2 \left(1 + 2 \frac{m_W^2}{m_t^2}\right) \quad (4.40)$$

As promised, this result grows as m_t^3 .

The result we have found is doubly surprising because the large result comes from the $q_\mu q_\nu$ terms in (4.37). In quantum electrodynamics, this term in the spin sum always cancels out due to current conservation. But in weak interaction theory, the current $\bar{b}\gamma^\mu(1 - \gamma^5)t$ which mediates the top decay is not conserved; in fact, its divergence is of order m_t .

However, in the context of our discussion of the interplay of gauge symmetry and Goldstone bosons, the result is easily understood. Though the quark charged current is not conserved, one can add terms involving Goldstone bosons to form the conserved gauge current of a spontaneously broken gauge theory. The analysis of the previous section applies to this theory directly. Thus, the Goldstone Boson Equivalence Theorem tells us that the leading behavior of the top quark width at high energy should be given by the Goldstone boson emission diagram of Fig. 8(b). I will now compute this diagram and verify the correspondence.

The matrix element for the emission of a Goldstone boson from a top quark, as shown in Fig. 8(b), is

$$i\mathcal{M} = -\lambda_t \bar{u}(p_b) \frac{(1 + \gamma^5)}{2} u(p_t) . \quad (4.41)$$

In this expression, λ_t is the coupling of the top quark to the Higgs boson. In the standard model, both the top quark mass and the W boson mass arise from the Higgs field vacuum expectation value v , according to the relations

$$m_t = \frac{\lambda_t v}{\sqrt{2}} , \quad m_W = \frac{gv}{2} . \quad (4.42)$$

Using these formulae to eliminate λ_t and v , we find

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = \lambda_t^2 p_b \cdot p_t = \frac{g^2 m_t^2}{2m_W^2} \cdot \frac{m_t^2}{2} \quad (4.43)$$

This leads to an expression for the top quark width

$$\Gamma_t = \frac{g^2}{64\pi} \frac{m_t^3}{m_W^2} \quad (4.44)$$

which does indeed capture the leading behavior of (4.40).

A more complex process which is strongly affected by the physics of massive spin 1 particles is the reaction $e^+e^- \rightarrow W^+W^-$. I will present a semiquantitative discussion of this reaction; the full tree-level formulae can be found, for example, in ref. 15.

The leading order diagrams contributing to $e^+e^- \rightarrow W^+W^-$ in the standard electroweak model are shown in Fig. 9(a). To understand the conceptual issues which this process addresses, let us make a naive estimate of the first of these diagrams. Roughly, we expect the amplitude for W pair production by a virtual photon to be given by the amplitude for production of charged scalars, times a polarization inner product:

$$i\mathcal{M}_\gamma \sim i\mathcal{M}(e^+e^- \rightarrow \phi^+\phi^-) \cdot \epsilon^{*\mu}(q_+) \epsilon_\mu^*(q_-) . \quad (4.45)$$

However, for the case of longitudinally polarized W bosons, the polarization product in (4.45) is very large. We can apply (4.33) to estimate

$$\epsilon^{*\mu}(q_+) \epsilon_\mu^*(q_-) \simeq \frac{q_+ \cdot q_-}{m_W^2} = \frac{s}{2m_W^2} . \quad (4.46)$$

This estimate of the matrix element of W pair production would imply

$$\frac{d\sigma}{d\cos\theta} \sim \frac{\pi\alpha}{s} \cdot \left(\frac{s}{2m_W^2}\right)^2 . \quad (4.47)$$

But this result is unphysically large. The $(1/s)$ behavior of the first factor in (4.47) is actually the largest asymptotic behavior allowed by unitarity for a single partial wave. Somehow, the strong energy dependence of (4.46) must be cancelled in the full result for the cross section.

Our general analysis of massive vector bosons tells us that this cancellation must occur, and that the matrix element for longitudinal W pair production must eventually be reduced to that for Goldstone boson pair production. I will sketch how this works in the amplitude for annihilation of polarized electrons, $\mathcal{M}(e_L^- e_R^+ \rightarrow W_{\text{long}}^+ W_{\text{long}}^-)$.

The first two diagrams in Fig. 9(a) have the same general form, in which the two W polarization vectors dot into the Yang-Mills vertex function. Using the approximation (4.33) for the longitudinal polarization vectors, one can simplify this vertex and arrive at the following expression:

$$i\mathcal{M}_{\gamma+Z} = -ie^2(q_+ - q_-)^\mu \bar{v}(p_+) \gamma_\mu \frac{(1 - \gamma^5)}{2} u(p_-) \cdot \left(\frac{1}{s} + \frac{\frac{1}{2} - \sin^2 \theta_w}{\sin^2 \theta_w} \frac{1}{s - m_Z^2} \right) \cdot \frac{s - 2m_w^2}{2m_W^2}. \quad (4.48)$$

In this expression, the first term is the matrix element for scalar boson production, the term in parentheses is the coherent sum of virtual photon and Z propagators, and the final term is the result of contracting longitudinal polarization vectors with the three-boson vertex. Though there is some cancellation between the photon and Z contributions, the full result still shows the pathology described in the previous paragraph.

The third diagram of Fig. 9(a) has a different kinematic structure. However, when one contracts this diagram with the longitudinal polarization vectors, one finds

$$i\mathcal{M}_\nu = -i \frac{e^2}{2 \sin^2 \theta_w} \bar{v}(p_+) \frac{\gamma \cdot q_+}{m_W} \frac{\gamma \cdot (p_- - q_-)}{(p_- - q_-)^2} \frac{\gamma \cdot q_-}{m_W} \frac{(1 - \gamma^5)}{2} u(p_-). \quad (4.49)$$

Since $p_- u(p_-) = 0$, we can replace $(\gamma \cdot q_-)$ by $\gamma \cdot (q_- - p_-)$; this factor cancels the neutrino propagator. Then the expression (4.49) can be rearranged into

$$i\mathcal{M}_\nu = -ie^2(q_+ - q_-)^\mu \bar{v}(p_+) \gamma_\mu \frac{(1 - \gamma^5)}{2} u(p_-) \cdot \left(-\frac{1}{2 \sin^2 \theta_w} \frac{1}{2m_W} \right) \quad (4.50)$$

Now add (4.48) and (4.50), take the high energy limit, and use the standard

model relation $m_W^2 = m_Z^2 \cos^2 \theta_w$. This gives

$$\begin{aligned}
i\mathcal{M} &\simeq -ie^2(q_+ - q_-)^\mu \bar{v}(p_+) \gamma_\mu \frac{(1 - \gamma^5)}{2} u(p_-) \\
&\quad \cdot \left(\frac{m_Z^2}{s - m_Z^2} \frac{\frac{1}{2} - \sin^2 \theta_w}{2m_W^2 \sin^2 \theta_w} - \frac{1}{s - M_Z^2} \frac{1}{2 \sin^2 \theta_w} \right) \\
&\simeq -ie^2(q_+ - q_-)^\mu \bar{v}(p_+) \gamma_\mu \frac{(1 - \gamma^5)}{2} u(p_-) \cdot \frac{1}{s} \cdot \left(\frac{1 - 2 \sin^2 \theta_w - 2 \cos^2 \theta_w}{4 \sin^2 \theta_w \cos^2 \theta_w} \right) \\
&\simeq +ie^2(q_+ - q_-)^\mu \bar{v}(p_+) \gamma_\mu \frac{(1 - \gamma^5)}{2} u(p_-) \cdot \frac{1}{s} \cdot \left(\frac{1}{4 \sin^2 \theta_w} + \frac{1}{4 \cos^2 \theta_w} \right). \tag{4.51}
\end{aligned}$$

This last answer is exactly the result of computing the diagrams of Fig. 9(b), in which the gauge bosons of the standard weak interaction model create pairs of Goldstone bosons. In the expression in parentheses in the last line, the first term, with the coupling $(e/\sin \theta_w)$, is the contribution of $SU(2)$ boson exchange, while the second term, with the coupling $(e/\cos \theta_w)$, is the contribution of $U(1)$ boson exchange. The final answer not only respects unitarity but actually is smaller than the amplitude for the pair production of transversely polarized W bosons. The cancellations that lead to this point are organized by the Goldstone Boson Equivalence Theorem and the underlying principle of exact local gauge invariance.

5. Higher Spin

We have now seen that the theory of spin 1 fields and their associated particles is surprisingly complex. In particular, it requires a higher principle such as current conservation to organize the states created by the field and to neatly cancel all contributions except those from physical propagating modes. These cancellations must occur even more strongly and more intricately in theories of spin greater than 1. I will now explain how our earlier arguments generalize to these cases. In this discussion, I will concentrate on theories of massless particles. As was

demonstrated in the previous section, the corresponding massive theories are built from the massless theories and are, if anything, more highly constrained.

In my general discussion of the connection between fields and particles, I pointed out that, as the spin of a field increases, fewer and fewer of its components create and destroy physical propagating states. In general, only the components of maximal helicity are physical. Thus, from the 8 complex-valued components of a spin $\frac{3}{2}$ field, only two— $\psi_{++\dot{\gamma}}$ and $\psi_{-\dot{-}\dot{\gamma}}$ —create and destroy physical particles moving in the $\hat{3}$ direction. All other states which are created by the various components of $\psi_{\alpha\dot{a}\dot{\beta}}$ in some mathematical formalism must be made to cancel out. This applies most strongly to the states of negative norm created by $\sigma^{0\alpha\dot{a}}\psi_{\alpha\dot{a}\dot{b}}$.

The cancellation of these unphysical components occurs naturally, just as in the spin 1 case, when the higher spin field couples to a conserved tensor. Thus, we can make a consistent theory of a massless spin $\frac{3}{2}$ field $\psi_{\mu\dot{\alpha}}$ in a theory which contains a conserved spin $\frac{3}{2}$ tensor current $s_{\mu\dot{\alpha}}$, with the cancellations of unphysical modes following from the pair of equations

$$\partial^\mu s_{\mu\dot{\alpha}} = 0 . \quad (5.1)$$

Similarly, we can construct a consistent theory of a massless spin 2 field $h_{\mu\nu}$ by coupling it to a conserved two-index tensor $t_{\mu\nu}$. In Yang-Mills theory, the coupling to the gauge field changes the current conservation equation $\partial^\mu j_\mu = 0$ to a modified, gauge-covariant equation $D^\mu j_\mu = 0$ which agrees with the standard equation to leading order. Such a modification is also typical in theories of higher spin.

To construct a theory of higher spin fields, we must thus ask, what conserved tensors of higher spin are available to be the sources of the new fields? One candidate is obvious. The energy-momentum tensor of all particles and fields, $T_{\mu\nu}$, is naturally conserved and can be considered as the source of a spin 2 field. The gauge theory of spin 2 which results from this coupling is precisely general relativity. The conservation law of the energy-momentum tensor is modified self-consistently

to

$$\nabla^\mu T_{\mu\nu} = 0 , \quad (5.2)$$

the analogue of the covariant conservation law for the Yang-Mills current. General relativity contains only helicity ± 2 particles as propagating states, and the production of these particles from the conserved energy-momentum tensor naturally cancels additional, unphysical modes.^[16,17]

In order to construct a higher-spin field in addition to the gravitational field, we must identify a second naturally conserved tensor. Unfortunately, this is extremely difficult; almost every possible case is excluded by general restrictions on the S -matrix proved by Coleman and Mandula.^[18] To understand the origin of these restrictions, let us consider the constraints on the existence of a second conserved two-index tensor $r_{\mu\nu}$, in addition to the full energy-momentum tensor $T_{\mu\nu}$.

The spatial integrals of $T_{\mu\nu}$ give a globally conserved energy-momentum 4-vector P^μ . Similarly, let us define the 4-vector

$$R^\mu = \int d^3x r^{0\mu} . \quad (5.3)$$

The vector R^μ is an additional conserved quantity which restricts scattering processes. By Lorentz covariance, the diagonal matrix elements of R^μ in one particle states of momentum p are proportional to p^μ ,

$$\langle p, a | R^\mu | p, a \rangle = C_a p^\mu , \quad (5.4)$$

where the constant of proportionality C_a depends only on the particle type. Now consider the elastic scattering of particles of two different types, $1 + 2 \rightarrow 1 + 2$. Conservation of P^μ yields the constraint

$$p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu . \quad (5.5)$$

Conservation of R^μ yields the additional equation

$$C_1 p_1^\mu + C_2 p_2^\mu = C_1 p_1'^\mu + C_2 p_2'^\mu . \quad (5.6)$$

The constraint (5.5) is solved by going to the center of mass frame; then (if the final state remains in the $\hat{1}-\hat{3}$ plane) the allowed values of \vec{p}'_1 and \vec{p}'_2 lie on a circle. This constraint is shown in Fig. 10. In this frame, the constraint (5.6) restricts the vector \vec{p}'_1 to an ellipse. The two constraints intersect for forward scattering and possibly at some additional specific angles. However the general property that the S -matrix is analytic in the momentum transfer t forbids such discrete solutions. We conclude that, if both conditions (5.5) and (5.6) are to be imposed simultaneously, there can be no elastic scattering of 1 from 2.

The theorem of Coleman and Mandula^[18] generalizes this argument to forbid additional conserved 4-vectors and conserved tensors of any higher rank (except for the Lorentz group generators $M_{\mu\nu}$). Thus, it implies that gravity is the only consistent theory of a spin 2 field, and that there are no consistent theories of massless fields with spin higher than 2. At least, no such theory can be constructed according to the strategy described here.

The case of spin $\frac{3}{2}$ is more ambiguous. A conserved spin $\frac{3}{2}$ current $s_{\mu\dot{a}}$ leads to a conserved spinor charge $Q_{\dot{a}}$. However, such a charge does not have diagonal matrix elements in single particle states. Thus, it is not necessarily forbidden, but its properties are strongly restricted. We will discuss this case in Section 7. For the case of spin $\frac{5}{2}$, the loophole available for spin $\frac{3}{2}$ can be closed, and the required source is forbidden by the Coleman-Mandula theorem. Thus, we come to the end of our catalogue of possible higher-spin fields.

6. Spin $\frac{1}{2}$ as a Construct

Now I will turn to the question with which we began these lectures: Why is there spin $\frac{1}{2}$? In this section and the next, I will describe two possible solutions. In this section, I will show how to construct spin $\frac{1}{2}$ from a mechanical model of particle dynamics. In the next section, I will describe a symmetry, which might be a fundamental symmetry of Nature, which requires spin $\frac{1}{2}$ to complete its symmetry multiplets.

6.1. A MODEL OF A SCALAR PARTICLE

Before we can form a mechanical model which produces relativistic spin $\frac{1}{2}$ particles, we should construct a model which leads to ordinary relativistic scalar particles. This is easily done by imagining scalar particles as point objects which move through space-time along world lines, and then write the mathematics appropriate to this physical picture. A natural guess is that the quantum mechanical amplitude for a particle to propagate from the space-time point y to x is given by an integral over paths

$$D(x, y) = \int \mathcal{D}X^\mu \exp(iS[X^\mu]) , \quad (6.1)$$

where $X^\mu(s)$ is a path from y to x $S[X]$ is some appropriate phase that the particle's wave function acquires as it moves along the path. A particle of mass m at rest would be expected to acquire a factor

$$e^{-imt} ; \quad (6.2)$$

thus, a reasonable guess for S is that it is proportional to the proper time which elapses along the path:

$$S = -m \int ds \sqrt{\left(\frac{dX^\mu}{ds}\right)^2} . \quad (6.3)$$

Does the expression (6.1) with (6.3) really lead to a description of relativistic scalar particles? It will be more straightforward to work with this expression if

we rewrite it in such a way that the square root in (6.3) is removed. To do this, introduce a new parameter $e(s)$ which is a function of the position on the path, and write

$$D(x, y) = \int \mathcal{D}X^\mu \mathcal{D}e \exp\left(-i \int ds \frac{1}{2} \left[\frac{(\dot{X})^2}{e} + em^2 \right]\right), \quad (6.4)$$

where $\dot{X}^\mu = dX^\mu/ds$. The variational equation for $e(s)$ is

$$-\frac{(\dot{X})^2}{e^2} + m^2 = 0; \quad (6.5)$$

fixing $e(s)$ as the solution to this equation, and substituting into the exponent of (6.4), we recover (6.3). Thus, (6.4) is also a reasonable starting point for our discussion.

In the construction of (6.3) and (6.4), position along the path is parametrized by the coordinate s . However, in both expressions for the path integral, the choice of the parameter s is arbitrary. Both exponents have the local invariance

$$X^\mu(s) \rightarrow X^\mu(g(s)), \quad e(s) \rightarrow \frac{dg}{ds} e(g(s)), \quad (6.6)$$

corresponding to the change of variables $s \rightarrow g(s)$. Since this is an invariance at each point s , it is a gauge symmetry of the path integrals, and these integrals must be defined by Fadde'ev-Popov gauge fixing. The gauge freedom of (6.4) can be fixed in a simple way: Set

$$e(s) = 1. \quad (6.7)$$

There are no ghosts or other awkward consequences of this choice of gauge.

With this prescription, the general sum over paths can be written as a sum over paths for which the parameter s runs from 0 to T , and an integral over T .

We thus obtain

$$D(x, y) = \int_0^\infty dT \int \mathcal{D}X^\mu \exp\left(-i \int ds \frac{1}{2}[(\dot{X})^2 + m^2]\right), \quad (6.8)$$

This functional integral can be evaluated explicitly. It is, in fact, the Feynman path integral for a nonrelativistic system with Hamiltonian

$$H = m^2 - p^2 = m^2 + (\vec{p})^2 - (p^0)^2, \quad (6.9)$$

integrated over time T :

$$\begin{aligned} D(x, y) &= \int_0^\infty dT \langle x | e^{-i(m^2-p^2)T} | y \rangle \\ &= \frac{i}{p^2 - m^2}. \end{aligned} \quad (6.10)$$

The final result is the Feynman propagator for a free scalar particle, the best result we could have hoped for.

Though we set up this construction by considering the scalar particle as an world line embedded in space-time, an alternative viewpoint is possible. We could as well interpret eq. (6.4) or (6.8) as representing an abstract world line, with a one-dimensional quantum field theory living on it. The Lagrangian of this quantum field theory is

$$L = -\frac{1}{2}[(\dot{X})^2 + m^2]. \quad (6.11)$$

In this view, the space-time coordinates $X^\mu(s)$ are fields which are a part of this one-dimensional field theory. In other words, one may view space-time as living *on* the world line as easily as one might view the world line as living *in* space-time.

6.2. ADDITION OF SPIN

From the point of view in which space-time is an attribute of the particle's world line, it is easy to find generalizations which produce more interesting types of particles. A simple way to modify (6.11) is to add an anticommuting coordinate field $\theta^\mu(s)$, $\mu = 0, 1, 2, 3$.^[19,20] The Lagrangian of this extended theory is

$$L = -\frac{1}{2}[(\dot{X})^2 + \theta^\mu \dot{\theta}_\mu + m^2] . \quad (6.12)$$

In principle, we might have tried adding any field that could live on the 1-dimensional world line. This particular choice, however, is especially interesting because the Lagrangian (6.12) has an unusual symmetry. Consider the transformation generated by

$$\delta X^\mu = \epsilon \theta^\mu , \quad \delta \theta^\mu = -\epsilon \dot{X}^\mu , \quad (6.13)$$

where ϵ is an anticommuting number: $\epsilon_1 \epsilon_2 = -\epsilon_2 \epsilon_1$. The second variation under this transformation is

$$(\delta_1 \delta_2 - \delta_2 \delta_1)A = 2\epsilon_2 \epsilon_1 \dot{A} , \quad (6.14)$$

where A is X^μ or θ^μ . Thus, the transformation (6.13) is in some sense the square root of a translation in s . It is not difficult to show that (6.13) is a symmetry of the Lagrangian (6.12):

$$\delta \int ds L = \int ds (\dot{X}^\mu \epsilon \dot{\theta}_\mu - \epsilon \dot{X}^\mu \dot{\theta}_\mu) = 0 . \quad (6.15)$$

For the moment, we might view this symmetry as an amusing feature of this particular extension; we will have more to say about it in Section 7.

In order to find the interpretation of (6.12) in terms of particles, we need to find the analogue of eq. (6.10) for the one-dimensional functional integral which contains this Lagrangian. The part of (6.12) which contains commuting numbers

is treated just as in (6.10). For the anticommuting numbers, the Lagrangian term $\frac{1}{2}\theta \cdot \dot{\theta}$ should be compared to the standard expression for commuting numbers

$$L = \sum_i p_i \dot{q}_i - H(p, q) . \quad (6.16)$$

To make the analogy, we take the half of the θ^μ to be canonical coordinates and the other half to be their conjugate momenta, with $H = 0$. Since these objects anticommute, we should convert them to quantum operators with anticommutation relations. Then the quantum operators θ^μ obey

$$\{\theta^\mu, \theta^\nu\} = -\frac{1}{2}g^{\mu\nu} . \quad (6.17)$$

One way to interpret these set of relations is to diagonalize it by finding two linear combinations of the θ^μ , which might be called a_i , and two orthogonal linear combinations a_i^\dagger which obey

$$\{a_i, a_j^\dagger\} = \delta_{ij} . \quad (6.18)$$

Then the Hilbert space acted on by the θ^μ is described by four states

$$|\phi\rangle , \quad a_1^\dagger |\phi\rangle , \quad a_2^\dagger |\phi\rangle , \quad a_1^\dagger a_2^\dagger |\phi\rangle . \quad (6.19)$$

Alternatively, we might recognize that the algebra (6.17) is exactly the Dirac algebra (2.25). Then the four states indicated schematically in (6.19) correspond to the four-dimensional Dirac representation of the Lorentz group. Thus, the particle which moves along the world line carries a Dirac spinor and thus has spin $\frac{1}{2}$.

This particular constructive picture of spin works uniquely for spin $\frac{1}{2}$. However, it is easily generalized to provide a construction of spin 1, and also higher spins. To construct particles with spin 1, choose the Lagrangian

$$L = -\frac{1}{2}[(\dot{X})^2 + m^2] - \bar{\theta}^\mu \dot{\theta}_\mu + \omega \bar{\theta}^\mu \theta_\mu , \quad (6.20)$$

where now θ^μ is a complex anticommuting number, with $\bar{\theta}^\mu$ its complex conjugate. Interpreting these variables as canonical coordinates and momenta according to

(6.16), we are led to the set of commutation relations

$$\{\bar{\theta}^\mu, \theta^\nu\} = -g^{\mu\nu} . \quad (6.21)$$

and the Hamiltonian

$$H = -\omega \bar{\theta} \cdot \theta . \quad (6.22)$$

Given (6.21), it is natural to interpret θ^μ as a set of fermionic annihilation operators a^μ (with positive metric for $\mu = 1, 2, 3$) and $\bar{\theta}^\mu$ as the corresponding creation operators $a^{\dagger\mu}$. The Hamiltonian is $H = \omega a^\dagger \cdot a$. Then this theory contains particle world lines associated with the various possible fermion states:

$$\begin{aligned} |0\rangle &\quad \text{spin 0 , } (\text{mass})^2 = m^2 \\ a^{\dagger i} |0\rangle &\quad \text{spin 1 , } (\text{mass})^2 = m^2 + \omega , \end{aligned} \quad (6.23)$$

and so on.

Though this analysis does give a construction of spin 1, it raises as many questions as it solves. For example, what kind of field is associated with $a^{\dagger i} a^{\dagger j} |0\rangle$? This field should be present in the theory unless it is removed by taking ω very large. There are problems in obtaining *massless* spin 1 particles: If $(m^2 + \omega) = 0$, then either scalar particles or the new states with two world-line fermions will have negative $(\text{mass})^2$. Finally, the construction contains explicit negative norm states $a^{\dagger 0} |0\rangle$. These states must have a mechanism to cancel completely from all scattering processes. It is possible to give satisfactory answers to these questions, but only within a formal structure more constraining than the particle models discussed in this section. We will find a better setting for addressing these questions in Section 8.

7. Spin $\frac{1}{2}$ as a Symmetry

As an alternative to explaining spin as a mechanical attribute of particles, one might attempt to postulate a symmetry of Nature which naturally leads to spin $\frac{1}{2}$. In this type of model, one would postulate that there exists an operator Q which generates a symmetry of the Hamiltonian of particle interactions and also has the property of converting particles of zero spin into particles with spin $\frac{1}{2}$. Such a symmetry is known as a ‘supersymmetry’. The first renormalizable field theory with a supersymmetry was constructed by Wess and Zumino.^[21] The formal consequences of supersymmetry are presented in detail in the book of Wess and Bagger.^[22]

7.1. THE SUPERSYMMETRY ALGEBRA

Any charge which converts spin 0 to spin $\frac{1}{2}$ must carry a spinor index. The simplest choice, and actually the only consistent one, is to take this charge to carry spin $\frac{1}{2}$. However, the hypothesis of a spin $\frac{1}{2}$ charge Q_α which commutes with the Hamiltonian turns out to be extremely restrictive. To see this, write the anticommutator of the charge Q_α with its Hermitian conjugate:

$$\{Q_\alpha, Q_{\dot{\alpha}}^\dagger\} = R_{\alpha\dot{\alpha}} \quad (7.1)$$

If Q commutes with the Hamiltonian, Q^\dagger will as well; thus $R_{\alpha\dot{\alpha}}$ commutes with the Hamiltonian. We recognize $R_{\alpha\dot{\alpha}}$ as a conserved vector, just the sort of object which was excluded by the Coleman-Mandula theorem, as described in the discussion following eq. (5.3). On the other hand, $R_{\alpha\dot{\alpha}}$ cannot be zero: Since R is the square of Q , $R = 0$ only if both Q and Q^\dagger give zero on all states of the Hilbert space.

There is only one way out of this dilemma. $R_{\alpha\dot{\alpha}}$ must be the one conserved 4-vector allowed by the Coleman-Mandula theorem—the total energy-momentum P^μ . Thus, if Q_α is a symmetry of the Hamiltonian carrying spin $\frac{1}{2}$, it must obey

the commutation relation

$$\{Q_\alpha, Q_{\dot{\alpha}}^\dagger\} = 2\sigma_{a\dot{\alpha}}^\mu P_\mu . \quad (7.2)$$

No half-measures are possible. Q_α must be a fundamental symmetry of space-time, generalizing the Poincaré algebra. The new symmetry generated by Q_α must act on every particle in Nature. Thus, we are led to a profound generalization of the theory of elementary particles.

To understand the consequences of the symmetry algebra (7.2) a bit better, consider the representation of this algebra in the simplest context—massless single-particle states moving in the $\hat{3}$ direction. For such states, we saw that in Section 2.5 that the Poincaré algebra has one-dimensional representations, plus their reflections under CPT . We will now analyze how the algebra (7.2) links these representations.

The first step in working with (7.2) is to write the $++$ and $--$ components of (7.2) explicitly:

$$\begin{aligned} \{Q_+, Q_+^\dagger\} &= 2(H - P^3) \\ \{Q_-, Q_-^\dagger\} &= 2(H + P^3) \end{aligned} \quad (7.3)$$

Since the quantities on the right-hand side generate translations in time and space, these commutation relations are reminiscent of (6.14). In fact, they represent the correct generalization of (6.14) to a multidimensional space-time.

A massless particle moving in the $\hat{3}$ direction satisfies $(H - P^3) = 0$, $(H + P^3) = 2P^3$. Thus, Q_+ and Q_+^\dagger must give zero on such states, while Q_- and Q_-^\dagger give a nonzero result. Define

$$\frac{1}{\sqrt{4P^3}}Q_- = a , \quad \frac{1}{\sqrt{4P^3}}Q_-^\dagger = a^\dagger . \quad (7.4)$$

Then the second line of (7.3) becomes

$$\{a, a^\dagger\} = 1 . \quad (7.5)$$

The operators (7.4) thus act on the one-particle states as fermion creation and annihilation operators. Since these operators raise and lower the $\hat{3}$ component of

angular momentum, and thus the helicity, we can view the pairs of states with and without the fermion created by a^\dagger as pairs of states $|p, \lambda\rangle$:

$$\begin{aligned} |p, 0\rangle &\leftrightarrow \left| p, \frac{1}{2} \right\rangle \\ \left| p, \frac{1}{2} \right\rangle &\leftrightarrow |p, 1\rangle , \end{aligned} \tag{7.6}$$

and so forth. These relations clarify the intuitive idea that supersymmetry links bosonic and fermion particle states.

7.2. SUPERSYMMETRIC DYNAMICS

The requirement of supersymmetry thus forces a quantum field theory to contain spin $\frac{1}{2}$ particles and fields. According to eq. (7.6), and its generalization to massive states, a spin zero particle in a supersymmetric theory must have a spin $\frac{1}{2}$ partner, and a spin 1 particle must have either a spin $\frac{1}{2}$ or a spin $\frac{3}{2}$ partner. The interactions of these new fermions are linked to the interactions of the bosons through the constraint of supersymmetry.

Perhaps the most interesting case from the viewpoint of first principles is the relation between spin 1 and spin $\frac{1}{2}$. Local gauge invariance requires the existence of spin 1 particles. Since supersymmetry in turn requires the existence of spin $\frac{1}{2}$ particles, it seems that we might construct a complete rationale for the particles which compose the standard model. However, the details do not fall into place correctly.

Given the existence of gauge bosons, supersymmetry specifies the quantum numbers and interactions of the new fermions. In the simplest realization of supersymmetry, the global symmetry charges commute with the supersymmetry charges:

$$[Q_A, Q_\alpha] = 0 . \tag{7.7}$$

We will see below that the strong constraints from this relation are not made weaker in more complicated realizations of the algebra. From (7.7), the partners

of a set of gauge bosons A_μ^A are fermions λ_α^A which belong to the same (adjoint) representation of the gauge group. As shown in Fig. 11(a), the Yang-Mills vertex for gauge bosons induces an interaction between the gauge boson and the new spin $\frac{1}{2}$ particle. This interaction is exactly the one present in the simple Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^A)^2 + \bar{\lambda} i\sigma \cdot D\lambda . \quad (7.8)$$

The Lagrangian (7.8) is the simple minimal coupling of a gauge particle to a chiral fermion, but it happens that, when the fermion and the gauge boson belong to the same representation of the gauge group, this Lagrangian is supersymmetric.

Unfortunately, the representation assignment of the fermions is precisely not what is needed to construct the standard model. From W bosons in the $I = 1$ representation of weak interaction $SU(2)$, supersymmetry would require $I = 1$ fermions \tilde{w}_α . From the gluons, which belong to the octet representation of color $SU(3)$, supersymmetry would require a multiplet of fermions \tilde{g}_a which also are color octet. These particles have no analogue in the standard model.

In fact, the restriction (7.7) makes it impossible to explain any fermion-boson correspondence seen so far in Nature. A left-handed quark q_L has as its supersymmetry partner a boson \tilde{q}_L with color 3, $I = \frac{1}{2}$, and hypercharge $Y = \frac{1}{6}$. The right-handed leptons ℓ_R or $\bar{\ell}_L$ lead to scalar particles with $I = 0$ and $Y = 1$. The $SU(2)$ doublets of left-handed charged leptons and neutrinos, such as $E_L = (\nu_e, e_L)$, lead to scalar particles with $I = \frac{1}{2}$, $Y = -\frac{1}{2}$. These quantum numbers are, curiously, those of the Higgs boson ϕ . But it is difficult to understand how lepton number could be conserved while ϕ obtains a vacuum expectation value in a scheme where ϕ is the partner of E_L .

Thus, if supersymmetry is the origin of spin $\frac{1}{2}$, one cannot extend this idea to explain the detailed content of the standard model. One must, in fact, postulate a new, undiscovered particle as the partner of each known particle of the standard model. However, there are good reasons to believe that Nature is, nevertheless, supersymmetric at its most fundamental level. If one believes in the grand

unification of the gauge interactions of the standard model at some very high momentum scale m_G , supersymmetry is the only known way to stabilize the relation $m_\phi \ll m_G$, and is the most natural explanation of the values of coupling constants observed at the Z^0 . These motivations for supersymmetry are reviewed in more detail in refs. 23 and 24. In the remainder of my discussion, I will use supersymmetry, quite independently of its phenomenological justification, as a powerful geometrical symmetry which organizes our conception of space-time.

I should remark that, just as the supersymmetry relation between spin 1 and spin $\frac{1}{2}$ particles generates the spin $\frac{1}{2}$ interactions, so the relation between spin $\frac{1}{2}$ and spin 0 generates a set of vertices for the spin 0 fields. These are shown in Fig. 11(b). The spin 0 partner of the lepton doublet E_L , for example, acquires a coupling both to the W boson and to its partner the \tilde{w} , and also a 4-scalar self-coupling.

7.3. SPIN $\frac{3}{2}$ AND HIGHER SUPERSYMMETRIES

From the viewpoint of the theory of spin, one important feature of supersymmetry is that it provides the missing ingredient in the discussion of spin $\frac{3}{2}$ particles given at the end of Section 5. We argued there that a consistent theory of spin $\frac{3}{2}$ requires a conserved spin $\frac{3}{2}$ current $s_{\mu\alpha}$. Such a current would be associated with a global charge

$$Q_\alpha = \int d^3x s_{0\alpha} . \quad (7.9)$$

We have now seen that such a global charge is consistent only if it is a supersymmetry charge; then a spin $\frac{3}{2}$ field can be included in a field theory only if it couples to the current of supersymmetry.

In such a structure, the spin $\frac{3}{2}$ field $\psi_{\mu\alpha}$ becomes the gauge field of supersymmetry, and the whole theory acquires a gauge symmetry

$$\delta\psi_{\mu\alpha} = D_\mu\epsilon_\alpha , \quad (7.10)$$

where $\epsilon_\alpha(x)$ is an anticommuting parameter which defines a local supersymmetry

transformation. The composition of supersymmetry transformations gives a translation, and so the field $\psi_{\mu\alpha}$ should naturally be associated with the gauge field of local translations, the gravitation field. Indeed, the massless spin $\frac{3}{2}$ particle created by $\psi_{\mu\alpha}$ is naturally paired with the graviton in a multiplet with the form of (7.6). The field $\psi_{\mu\alpha}$ is then called the *gravitino*, and the resulting generalization of general relativity is called *supergravity*. The Lagrangian which includes gravitation and the natural coupling of gravity to $\psi_{\mu\alpha}$,

$$\mathcal{L} = \sqrt{-g}R + \frac{1}{2}\bar{\psi}_\mu i\epsilon^{\mu\nu\lambda\sigma}\gamma_\nu D_\lambda\psi_\sigma , \quad (7.11)$$

can be shown to be invariant under local supersymmetry transformations which link the fields $g_{\mu\nu}$ and $\psi_{\mu\alpha}$.^[25]

In this line of argument, it seems that there could be at most one spin $\frac{3}{2}$ field, just as there can be at most one graviton. However, more general supersymmetry algebras than (7.2) are possible, and by incorporating them, we may build larger theories. But this extension also brings in new restrictions, since the higher supersymmetry algebras imply still stronger relations among the couplings of the model.

The general restrictions of the Coleman-Mandula theorem on the presence of spin $\frac{1}{2}$ changes were worked out by Haag, Lopuszanski, and Sohnius.^[26] These authors showed that, although the restriction we found on the right hand side of (7.2) is absolute, more general theories can be built by incorporating several supersymmetry charges, each of which has a square which is the total energy-momentum. More explicitly, they allow a set of commutation relations

$$\{Q_\alpha^i, Q_{\dot{\alpha}}^{j\dagger}\} = 2\delta^{ij}\sigma_{a\dot{\alpha}}^\mu P_\mu . \quad (7.12)$$

This structure is known as *N-extended* supersymmetry. A theory with N supersymmetry charges can be gauged with one graviton and N gravitinos. The each of the various supersymmetry charges pair one gravitino with the graviton and the

others with spin 1 bosons which must also be present in the theory. These spin 1 bosons provide a gauge group which does not commute with the supersymmetry charges Q_α^i , providing the generalization of (7.7). We can use (7.12) to determine the exact particle content of theories with this higher symmetry.

To analyze the implications of (7.12), consider once again the action of the supersymmetry generators on massless one-particle states moving in the $\hat{3}$ direction. As in the paragraph below (7.3), we can convert the supersymmetry commutation relation

$$\{Q_-^i, Q_-^{j\dagger}\} = 2\delta^{ij}(H + P^3) \quad (7.13)$$

to a set of relations for fermion create and annihilation operators. Define

$$\frac{1}{\sqrt{4P^3}}Q_-^i = a^i, \quad \frac{1}{\sqrt{4P^3}}Q_-^{j\dagger} = a^{j\dagger}. \quad (7.14)$$

The operators a^i are helicity raising operators. Then the commutation relations (7.13) become

$$\{a^i, a^{j\dagger}\} = \delta^{ij}. \quad (7.15)$$

These operators build up a multiplet of 2^N states connected by extended supersymmetry.

The simplest examples of these multiplets occur in theories of $N = 2$ extended supersymmetry. The simplest representation, which is built on a state $|p, 0\rangle$ of helicity zero, is

$$\begin{pmatrix} & a^1 |p, 0\rangle & \\ |p, 0\rangle & & a^1 a^2 |p, 0\rangle \\ & a^2 |p, 0\rangle & \end{pmatrix} = \begin{pmatrix} & |p, \frac{1}{2}\rangle & \\ |p, 0\rangle & & |p, 1\rangle \\ & |p, \frac{1}{2}\rangle & \end{pmatrix}. \quad (7.16)$$

This multiplet contains a gauge boson, two chiral fermions, and a scalar: $(\phi^A, \lambda_\alpha^{1A}, \lambda_\alpha^{2A}, A_\mu^A)$.

The $N = 2$ multiplet with maximum helicity 2 is

$$\begin{pmatrix} & |p, \frac{3}{2}\rangle \\ |p, 1\rangle & & |p, 2\rangle \\ & |p, \frac{3}{2}\rangle \end{pmatrix}, \quad (7.17)$$

which contains a vector boson, two gravitinos, and the gravitons. The vector boson may be thought to generate a gauge symmetry unified with gravity. However, in this case, the symmetry is only $U(1)$. To construct higher symmetries within the supersymmetry multiplets, we must go to higher N .

For $N = 4$, one finds a multiplet

$$|p, -1\rangle \leftrightarrow 4 \times \left| p, -\frac{1}{2} \right\rangle \leftrightarrow 6 \times |p, 0\rangle \leftrightarrow 4 \times \left| p, \frac{1}{2} \right\rangle \leftrightarrow |p, 1\rangle . \quad (7.18)$$

This multiplet is CPT self-conjugate, it contains one vector boson, 4 chiral fermions, and 6 real scalar bosons. This is the largest multiplet for which all fields have spin less than or equal to 1. The field theory of this multiplet turns out to be quite magical; for example, its renormalization group β function vanishes to all orders in perturbation theory.^[27]

One might similarly ask for which values of N one finds a multiplet with a single graviton and no spin higher than 2. The largest such multiplet occurs for $N = 8$:

$$\begin{aligned} |p, -2\rangle &\leftrightarrow 8 \times \left| p, -\frac{3}{2} \right\rangle \leftrightarrow 28 \times |p, -1\rangle \leftrightarrow 56 \times \left| p, -\frac{1}{2} \right\rangle \leftrightarrow 70 \times |p, 0\rangle \\ &\leftrightarrow 56 \times \left| p, \frac{1}{2} \right\rangle \leftrightarrow 28 \times |p, 1\rangle \leftrightarrow 8 \times \left| p, \frac{3}{2} \right\rangle \leftrightarrow |p, 2\rangle . \end{aligned} \quad (7.19)$$

The multiplet contains 28 gauge bosons. These form the antisymmetric tensor representation of $SO(8)$, which is also the adjoint representation. Thus, this theory naturally contains a unified $SO(8)$ gauge theory. Unfortunately, this group is not large enough to contain the standard model gauge group $SU(3) \times SU(2) \times U(1)$.

Worse, this $SO(8)$ has vector-like couplings rather than the chiral couplings which are essential to build weak interaction theory. Cremmer and Julia^[28] have worked out the detailed structure of the $N = 8$ supergravity theory and have identified a large global symmetry group—a noncompact E_8 —but so far no one has succeeded in building a relation between this group and the gauge group of the standard model.

Thus, the idea that spin $\frac{1}{2}$ arises as the result of a symmetry of Nature turns out to be a very powerful one, but one which stops short of providing a complete theory of the fundamental interactions. To build more successful models, we need to add to supersymmetry some structure of a quite different kind.

8. A Fruitful Blend

One of the most remarkable theoretical developments of the past ten years has been the realization that it is possible to merge the ideas of the previous two sections in a fruitful way. In Section 6, we studied the idea of building a particle theory by putting a one-dimensional quantum field theory on a world line. It is not hard to imagine a generalization in which one imagines a particle as a line or ring which sweeps out a two-dimensional surface in space-time. In this context, we could build a particle model by putting a two-dimensional quantum field theory on this surface. It is not so obvious why this would give an improvement over the picture we found for world-line theories, or, on the other hand, why we should stop at two-dimensions rather than studying three- or four-dimensional objects embedded in space-time. I can only say that the two-dimensional case offers just the right balance between freedom and constraints to allow one to create theories with an intriguing amount of structure. This particular case is known as *string theory*, or, with the inclusion of supersymmetry, *superstring theory*.

8.1. THE BOSONIC STRING

The simplest sort of string theory is one in which a particle is a ring moving through space-time. The physical picture of particle motion is that shown on the right-hand side of Fig. 12: The particle sweeps out a world-surface in the form of a tube. This surface may split or branch, and these branches represent particle interactions. All of the properties of the particles described by this motion follow from a quantum field theory on two-dimensional surface shown on the left.

In the world-line theory of Section 6.1, we began our discussion by considering the world-line to be embedded in space-time. If we take a similar point of view here, we would describe the string dynamics by field $X^\mu(s, t)$. The arguments of the field (s, t) are coordinates on this surface. With a convenient choice of gauge, the $X^\mu(s, t)$ are free fields whose Lagrangian is

$$\int \mathcal{L} = \frac{T}{8\pi} \int ds dt [\partial_\lambda X^\mu \partial^\lambda X_\mu] , \quad (8.1)$$

with $\lambda = 0, 1$. The parameter T has the dimensions of $(\text{mass})^2$ and provides a natural length scale for the theory.

Now we can switch our perspective and regard space-time X^μ as a set of fields which lives on the string. Choose the coordinate s to run from 0 to 2π around the ring. Then $X^\mu(s)$ has the Fourier decomposition

$$X^\mu(s) = X_0^\mu + \sum_{n \neq 0} e^{ins} X_n^\mu . \quad (8.2)$$

Then X_0^μ is the center of mass position of the string, conjugate to the total 4-momentum P^μ . The X_n^μ are the coordinates of string oscillations. These can be quantized as harmonic oscillators corresponding to running waves moving to the left and to the right around the ring. Let a_n^μ and \bar{a}_n^μ , $n > 0$, be the annihilation operators corresponding to these two sets of harmonic oscillators. Then the dynamics

of the string is neatly captured in the formula for the energies of the various string states. This is the relativistic mass formula $P^2 = m^2$, with

$$m^2 = 2\pi T \left\{ \sum_n n(a_n^{\dagger\mu} a_n^\mu + \bar{a}_n^{\dagger\mu} \bar{a}_n^\mu) + 2\mathcal{Z} \right\}. \quad (8.3)$$

The offset $2\mathcal{Z}$ is the (renormalized) zero-point energy of the two sets of oscillators. The relation (8.3) between the oscillator excitations and the masses of string states clarifies that the system is at the same time relativistic and harmonic.

The mass formula (8.3) is reminiscent of the mass formula in (6.23) for the particle theory with spin 1. However, the two-dimensional field theory substructure provides three further restrictions. The first restriction is relatively simple: The level of excitation in the left-moving oscillators must be equal to the level of excitation in the right-moving oscillators. Otherwise, the coordinate system would rotate around the ring. The second restriction is a bizarre one: The number of dimensions of the space-time in which the string is embedded must be 26. I will explain below how this restriction may be relaxed. The third restriction precisely fixes the zero point energy \mathcal{Z} to the value (-1) .

Ignoring, for the moment, the strange second requirement, we can work out the lowest mass states in the spectrum of the string. The ground state is

$$|0\rangle, \quad m^2 = -4\pi T. \quad (8.4)$$

This state is an unphysical scalar tachyon, and this also must be eliminated in an improved theory. The first excited states are

$$a_1^{\dagger\mu} |0\rangle, \quad \bar{a}_1^{\dagger\mu} |0\rangle, \quad m^2 = -2\pi T. \quad (8.5)$$

However, these states are eliminated by the requirement that the left- and right-moving oscillators have the same degree of excitation. Thus, the next physical state of the theory is

$$a_1^{\dagger\mu} \bar{a}^{\dagger\nu} |0\rangle, \quad m^2 = 0. \quad (8.6)$$

This multiplet contains an exactly massless spin 2 particle—the graviton.

From the discussion of Section 5, we ought to be suspicious that this spin 2 particle is defined consistently, so that its unphysical components are not produced in scattering processes. However, in the string theory, one can prove that the unphysical production amplitudes naturally cancel. The strategy of this proof is geometrical, and is illustrated in Fig. 13. The emission of any string state occurs through a world-surface of the form shown on the left of the figure, with a long pipe branching off of the main surface. If we deform the pipe to be very long and thin, we can replace the pipe with a pointlike perturbation of the world surface, which can be represented by a local operator. Each particular particle state of the string corresponds to a different boundary condition at the end of the pipe, and therefore, through this construction, to a different local operator. For the particle state (8.6), the corresponding local operator is the energy-momentum tensor $T^{\mu\nu}(s, t)$ on the world surface. This is a conserved tensor, and so the unphysical components of the graviton cancel out naturally.

Actually, this argument reveals only a part of the deeper substructure of string theory. In higher dimensions, as we have seen, the energy-momentum tensor is the highest rank conserved tensor possible. However, in two dimensions it is possible to have extremely large geometrical symmetry groups and correspondingly high-rank conserved tensors. The special symmetry involved is familiar from the theory of classical partial differential equations: In two dimensions, equations which do not have an intrinsic scale, such as the Laplace equation, are solvable by conformal mapping. Under certain conditions, this symmetry of conformal mapping survives into the quantum theory. For free fields, these conditions precisely restrict the total number of fields in the theory and the zero-point energy, in just the manner described below (8.3). Since a surface branching into a long thin tube is related to a surface with a small hole by a conformal transformation, the relation shown in Fig. 13 requires no approximation and applies to every possible string state. This means that the interactions of a string are uniquely determined by its spectrum, a profound generalization of the constraints of gauge invariance. Similarly, the cancellation of unphysical states that we found for the graviton gen-

eralizes to the full string spectrum: By using the relation between the higher states of the string spectrum and the higher conserved tensors of the two-dimensional theory, one can show that all negative norm excitations created by the operators $a_n^{\dagger 0}$ cancel out of scattering matrix elements.^[29]

8.2. DECORATED STRINGS

The theory in which only the space-time coordinates $X^\mu(s, t)$ live on the world surface has some wonderful mathematical properties but also contains awkward features—a tachyonic particle and the restriction to 26 dimensions. To ameliorate these problems, we might try to put a different two-dimensional field theory onto the string world surface. Following the approach of Section 5.2, we can add anticommuting coordinate $\psi^\mu(s, t)$. In order for the negative norm excitations created by ψ^0 to be cancelled, we require a higher symmetry which incorporates both conformal invariance and supersymmetry. Thus, to add spin to the string in the manner of Section 5.2, we must already add two-dimensional supersymmetry. What do we get back in return?

Comparing this construction to that in Section 5.2, we see one new feature: The field ψ^μ is a function of the coordinate s which runs around the ring, and we must fix the boundary condition to be imposed on these fields. The simplest choice is periodic boundary conditions. With this choice, the zero point energy turns out to be $\mathcal{Z} = 0$ and so the $n = 0$ Fourier components of the ψ^μ link a multiplet of massless particles. In the quantum theory of the string, the operators corresponding to these modes satisfy

$$\{\psi_0^\mu, \psi_0^\nu\} = -\frac{1}{2}g^{\mu\nu}, \quad (8.7)$$

just as we found for the one-dimensional case in (6.17). The physical interpretation is the same: These strings are Dirac fermions in space-time. In the discussion below, I will denote states of this Dirac fermion multiplet as $|\alpha\rangle$.

In the same string theory, it is consistent to have other strings for which the boundary condition on $\psi^\mu(s)$ is different. One may consider, for example, antiperiodic boundary conditions: $\psi^\mu(2\pi) = -\psi^\mu(0)$. Now the Fourier expansion of $\psi^\mu(s)$ has the form

$$\psi^\mu(s) = \sum_{n \geq 0} e^{i(n+\frac{1}{2})s} \psi_{n+\frac{1}{2}}^\mu + c.c. \quad (8.8)$$

The zero point energy for these strings is $\mathcal{Z} = -\frac{1}{2}$, so the states

$$\psi_{\frac{1}{2}}^{\dagger\mu} \bar{\psi}_{\frac{1}{2}}^{\dagger\mu} |0\rangle \quad (8.9)$$

form a multiplet of massless particles which include the graviton. In fact, $\psi_{\frac{1}{2}}^\mu$ and $\psi_{\frac{1}{2}}^{\dagger\mu}$ have the same operator relations as the operators θ^μ and $\bar{\theta}^\mu$ in (6.22). Remarkably, string theory allows particles acted on by these operator to coexists with particles acted on by (8.7). The price of this coexistence is equally remarkable; it is that states with an even number of fermions in either the left- or right-moving sector cancel out of the S -matrix. This removes the tachyon $|0\rangle$, and also converts the multiplet $|\alpha\rangle$ acted on by (8.7) from a Dirac to a chiral fermion.

For either or both choices of boundary condition, the constraint on the total number of fields X^μ and ψ^μ is that μ should run over 10 dimensions. However, it is possible to lower this number to 4 dimensions, or any other convenient value, by decorating the string world surface with more free fields, or with interacting fields which satisfy the constraints needed for conformal invariance. The zero-point energy depends on the detailed collection of fields, and may be different for the left- and right-moving sectors. For example, one can build a theory with additional right-moving antiperiodic fermions $\bar{\chi}^i$, $i = 1, \dots, n$, in such a way that the total zero point energy is $\mathcal{Z} = -\frac{1}{2}$ from the left-moving sector and $\bar{\mathcal{Z}} = -1$ from the right-moving sector. Then the state

$$\psi_{\frac{1}{2}}^{\dagger\mu} \bar{\chi}_{\frac{1}{2}}^{\dagger i} \bar{\chi}_{\frac{1}{2}}^{\dagger j} |0\rangle \quad (8.10)$$

is an allowed particle of the theory. This state is massless, has spin 1, and transforms as an antisymmetric tensor of $SO(n)$. It is, in fact, an $SO(n)$ gauge boson.

If the content of the string theory is properly arranged, states which acquire a vector character from a left-moving excitation created by $\psi_{\frac{1}{2}}^{\dagger\mu}$ can be naturally paired with states for which the left-moving sector is a spinor state $|\alpha\rangle$. For example, the same theory which contains the state (8.9) will also contain

$$\bar{\psi}_{\frac{1}{2}}^{\dagger\mu} |\alpha\rangle . \quad (8.11)$$

This particle carries spin $\frac{3}{2}$ and is, in fact, the gravitino partner of the graviton given above. Similarly, the same theory which contains the state (8.10) will also contain

$$\bar{\chi}_{\frac{1}{2}}^{\dagger i} \bar{\chi}_{\frac{1}{2}}^{\dagger j} |\alpha\rangle , \quad (8.12)$$

the spin $\frac{1}{2}$ supersymmetry partner of the gauge boson. In this way, it is straightforward to construct string theories which have a supersymmetric spectrum, and, by extension, a full set of supersymmetric interactions.

From this point, the possibilities are limited only by one's imagination in assembling two-dimensional field theories with which to decorate the string world surface. A large number of model-building strategies for string theory are described in refs. 30–32. Using these strategies, one can build a wide variety of theories, some of which might even resemble the standard model. All of these theories, or at least all interesting ones, require spin $\frac{1}{2}$ or some generalization as an input to build the two-dimensional theory, but then recover spin $\frac{1}{2}$, and often supersymmetry as well, in its spectrum of particles in space-time.

9. Conclusions

Though we have found no definite answer to the question posed in the introduction, we have found ourselves led through a wonderful tangle of speculations on the deep structure of Nature. Is spin constructed or is it fundamental? Is it the requirement of symmetry? In the furthest flights we have taken, it seems that space-time itself is too restrictive a notion, and that we must generalize this in order to gain a full appreciation of spin. In any case, there is no doubt that spin must play a central role in unlocking the mysteries of fundamental physics.

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FIGURE CAPTIONS

- 1) The process which changes the helicity of a massive particle.
- 2) The minimal representation of the Poincaré group in the case of a massless particle.
- 3) Kinematics of the process $e^+e^- \rightarrow \mu^+\mu^-$.
- 4) Single photon emission in quantum electrodynamics.
- 5) Vector boson self-energy in the Higgs model (4.16).
- 6) Emission of a single massive vector boson.
- 7) The Goldstone Boson Equivalence Theorem.
- 8) Diagrams contributing to the top quark width: (a) the leading order contribution in the standard model; (b) the analogous Goldstone boson diagram.
- 9) Diagrams contributing to $e^+e^- \rightarrow W^+W^-$: (a) the leading order contributions in the standard model; (b) the analogous Goldstone boson diagrams.
- 10) Constraints on elastic scattering imposed by conservation of two 4-vectors P^μ and R^μ .
- 11) Supersymmetry specifies the interactions of fermions from the interactions of the corresponding bosons, or vice versa: (a) the interaction vertex for the spin $\frac{1}{2}$ partner of a gauge boson, (b) the interaction vertices for the spin 0 partner of the left-handed electron.
- 12) The dynamics of a periodically connected two-dimensional surface is viewed as a particle moving and interacting in space-time.
- 13) Relation between the amplitude for emission of a string state and an operator expectation value in the world-surface.